Test #3 100 points Math 200 Name:

Show work.

1) Find and classify all the critical points of $f(x,y)=3y+\left(x^2-4x\right)\ln(y)$. (10 points)

$y \xrightarrow{x=5} \infty,\ f \to \infty$

As $y \xrightarrow{x=5} 0^+,\ f \to -\infty$, so no global extremum

Find:

$f_x=(2x-4)\ln(y)=0 \Rightarrow x=2$ or $y=1$

$f_y=3+\frac{x^2-4x}{y}=0$, so

$x=2 \Rightarrow 3+\frac{(-4)}{y}=0 \Rightarrow y=\frac{4}{3}$

$y=1 \Rightarrow 3+x^2-4x=0$

$\Rightarrow x^2-4x+3=0 \Rightarrow (x-3)(x-1)=0$

$\Rightarrow x=+3,+1.$

$\therefore$ the c.p.'s are $(2,\frac{4}{3}), (+3,1), (+1,1)$

Classify:

$f_{xx}=2\ln(y)$; $f_{yy}=\frac{4x-x^2}{y^2}$

$f_{xy}=\frac{2x-4}{y}$.

$f_{xx}(2,\frac{4}{3})=2\ln(\frac{4}{3})>0$

and $|H|=2\ln(\frac{4}{3})\cdot\frac{4}{\frac{16}{9}}-0>0$

$\Rightarrow$ $(2,\frac{4}{3})$ is a local minimum

$f_{xx}(+3,1)=0$, $|H|=-100<0$

$\Rightarrow$ $(+3,1)$ is a saddle

$|H(+1,1)|=-36<0 \Rightarrow$ $(+1,1)$ is a saddle

2) Find and classify the extreme values of $f(x,y)=2x^2+8y^2$ for all points that lie on $xy=8$. **Sketch** a picture of level curves to verify your answer. (10 points)

$C=xy$; $\nabla f=\lambda\nabla C$

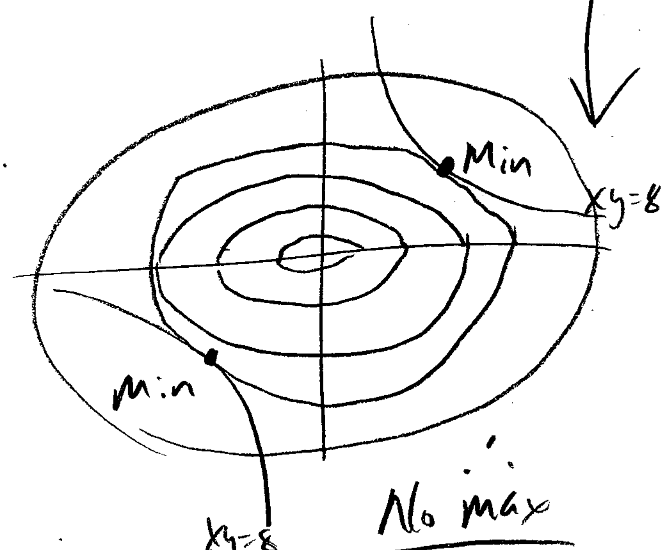
$\Rightarrow \begin{cases} 4x=\lambda y \\ 16y=\lambda x \end{cases} \Rightarrow \frac{x}{4y}=\frac{y}{x}$

$xy=8 \Rightarrow$ x and y are non zero with the same sign

$\Rightarrow 4y^2=x^2 \Rightarrow \pm 2y=x \Rightarrow 2y=x.$

$xy=8$ and $2y=x \Rightarrow 2y^2=8 \Rightarrow y=\pm 2.$

$\therefore$ the only constrained critical points are $(4,2), (-4,-2)$.



No max and two global mins.

$\therefore$ $(4,2), (-4,-2)$ are constrained global mins

3) Use the divergence theorem to find the flux of $\vec{F}(x,y,z)=\langle x^2-y, z^2, 3z+x\rangle$ out of the cube with side length of ~~two~~ four that is centered at (0, 0, 0). (10 points)

$$\Phi = \iint_{\substack{\text{Cube}\\ \parallel \\ Bd(E)}} \vec{F}\cdot d\vec{S} \underset{\substack{\text{Div}\\ \text{thm}}}{=} \iiint_{E=\substack{\text{Solid}\\ \text{Cube}}} \nabla\cdot\vec{F}\, dV$$

$$= \iiint_E 2x+0+3\, dV = \int_{-2}^{2}\int_{-2}^{2}\int_{-2}^{2} \overset{0;\ \text{ODD}}{\cancel{2x}}\, dx\,dy\,dz + 3\cdot \text{Vol}(E)$$

$$= 3\cdot 4^3 = \boxed{192} \text{ flux units}$$

4) Use Stokes' theorem to find the flux of $\nabla\times\vec{F}$ if $\vec{F}(x,y,z)=\langle x^2-y, z^2, 3z+x\rangle$ through the hemisphere $z=\sqrt{1-x^2-y^2}$ that is oriented with nonnegative z-coordinates. Note: there is no bottom to the hemisphere. (10 points)

$Bd(S)$: $x^2+y^2=1$ oriented positively :

$$\Phi = \iint_S \nabla\times\vec{F}\cdot d\vec{S} \underset{S.T.}{=} \oint_{Bd(S)} \vec{F}\cdot d\vec{s}$$

$\vec{r}(t) = \langle \cos t, \sin t, 0\rangle$

$$= \int_0^{2\pi} \langle \cos^2 t - \sin t, 0, \cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} -\sin t\cos^2 t + \sin^2 t\, dt = \frac{\cos^3 t}{3}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1-\overset{0;\ 2\pi}{\cancel{\cos(2t)}}}{2}\, dt$$

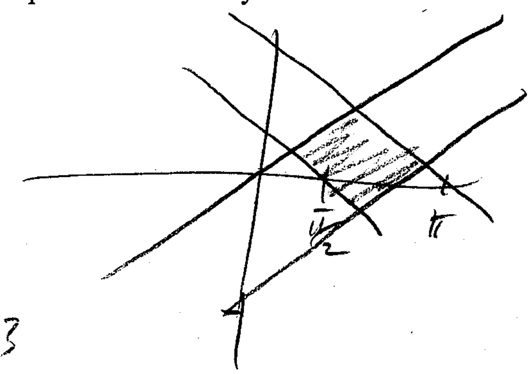
$$= \pi$$

You may use Stokes' theorem twice: $(S_1: z=0,\ x^2+y^2\le 1)$

OR $\Phi = \iint_S \nabla\times\vec{F}\cdot d\vec{S} \underset{S.T}{=} \int_{Bd(S)} \vec{F}\cdot d\vec{s} \underset{S.T.}{=} \iint_{S_1} \nabla\times\vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle -2z,-1,1\rangle\cdot\langle 0,0,1\rangle\, dA$

$= \text{Area}(x^2+y^2\le 1) = \pi$

5) Use a substitution to find $I = \iint_R \cos(x+y)\, dA$ if R is the region in the xy – plane bounded by $x+y=\frac{\pi}{2}$, $x+y=\pi$, $x-y=0$, and $x-y=3$. (10 points)



$$u = x+y,\quad v = x-y \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}, \text{ so } I = \int_{\pi/2}^{\pi}\int_0^3 \cos(u)\cdot\frac{1}{2}\, dv\, du$$

$$\Rightarrow I = \frac{3}{2}\int_{\pi/2}^{\pi} \cos(u)\, du = \frac{3}{2}\left(\sin(u)\right|_{\pi/2}^{\pi} = \boxed{-\frac{3}{2}}$$

6) Find $I = \iint_S \vec{F}(x,y,z)\cdot d\vec{S}$ if S is oriented up and parameterized by $\vec{r}(u,\theta) = \langle u\cos\theta, u\sin\theta, \sin(u^2)\rangle$, $\sqrt{\frac{\pi}{2}} \le u \le \sqrt{\pi}$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$, and $\vec{F}(x,y,z) = \left\langle \frac{2}{x}, \frac{2}{y}, 0\right\rangle$. (10 points)

$$\vec{r}_u = \langle \cos\theta, \sin\theta, 2u\cos(u^2)\rangle\, du$$

$$\vec{r}_\theta = \langle -u\sin\theta, u\cos\theta, 0\rangle\, d\theta$$

$$d\vec{S} = +\langle -2u^2\cos\theta\cos(u^2), -2u^2\sin\theta\cos(u^2), \underset{u>0}{u}\rangle\, d\theta\, du$$

$$\therefore I = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\int_{\pi/6}^{\pi/3} \left\langle \frac{2}{u\cos\theta}, \frac{2}{u\sin\theta}, 0\right\rangle \cdot d\vec{S}$$

$$= \int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\int_{\pi/6}^{\pi/3} -4u\cos(u^2) - 4u\cos(u^2)\, d\theta\, du$$

$$I = \left(\frac{\pi}{3} - \frac{\pi}{6}\right)\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} -8u\cos(u^2)\, du$$

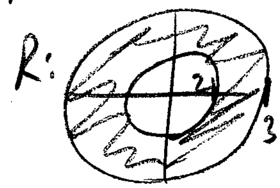
$$= \frac{\pi}{6}(-4)\sin(u^2)\Big|_{\sqrt{\pi/2}}^{\sqrt{\pi}}$$

$$= -\frac{2\pi}{3}(0-1)$$

$$= \boxed{\frac{2\pi}{3}}$$

7) Find $I = \iint_S \vec{F}(x,y,z)\cdot d\vec{S}$ if S is the piece of $z = y^2 + 5$, oriented up, such that ~~$-2 \le x \le 1$ and $0 \le y \le 1$~~, and $1 \le x^2+y^2 \le 9$

$\vec{F}(x,y,z) = \langle 0, -y, z\rangle$. (10 points)



$$z = y^2+5 \Rightarrow d\vec{S} = +\langle 0,-2y,1\rangle\,dx\,dy$$

$$\Rightarrow \vec{F}\cdot d\vec{S} = (2y^2+z)\,dx\,dy.$$

$$\therefore I = \iint_R 2y^2 + (y^2+5)\,dy\,dx$$

$$= \int_0^{2\pi}\int_1^3 3r^3\sin^2\theta\,dr\,d\theta + 5\cdot(\text{Area } R)$$

$$= \int_0^{2\pi} \frac{1-\cancel{\cos(2\theta)}^{\,0;\,2\pi}}{2}\,d\theta \cdot \left.\frac{3r^4}{4}\right|_1^3 + 5\pi(9-1)$$

$$I = \frac{3\pi}{4}(81-1) + 40\pi$$

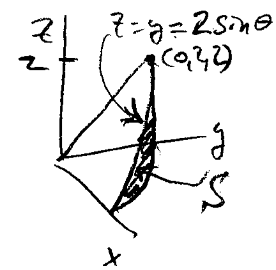
$$I = 60\pi + 40\pi$$

$$I = 100\pi$$

8) Calculate $I = \iint_S z\,dS$ if S is the piece of $x^2 + y^2 = 4$ in the **first octant** that lies below the plane z = y.

cylinder; special case.

(10 points)



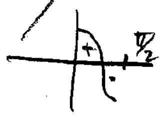
$$dS = 2\,dz\,d\theta$$

$$I = \int_0^{\pi/2}\int_0^{2\sin\theta} z\cdot 2\,dz\,d\theta$$

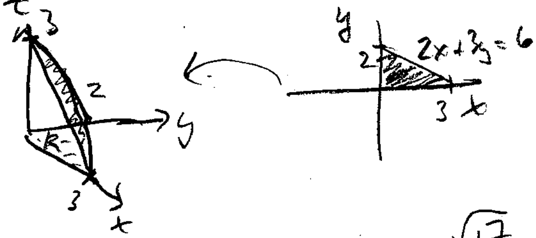
$$= \int_0^{\pi/2} \left(z^2\Big|_0^{2\sin\theta}\right) d\theta$$

$$= \int_0^{\pi/2} 4\sin^2\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 1-\cancel{\cos(2\theta)}^{\,0}\,d\theta = \pi$$



9) Find the mass of the part of the plane $2x+3y+2z=6$ that lies in the first octant if the density is $\delta(x,y,z)=y$ grams per square meter. (10 points)



$$z=\frac{6-2x-3y}{2}$$

$$\Rightarrow \vec{dS}=\pm\langle 1,\tfrac{3}{2},1\rangle\,dx\,dy$$

$$\Rightarrow dS=\sqrt{1+\tfrac{9}{4}+1}\,dx\,dy.$$

$$\text{Mass}=\iint_R y\frac{\sqrt{17}}{2}\,dx\,dy$$

$$=\frac{\sqrt{17}}{2}\int_0^2 y\left(\int_0^{\frac{6-3y}{2}}dx\right)dy$$

$$\text{Mass}=\frac{\sqrt{17}}{2}\int_0^2\frac{6y-3y^2}{2}\,dy$$

$$=\frac{\sqrt{17}}{4}\left(3y^2-y^3\right)\Big|_0^2=\boxed{\sqrt{17}}\ \underline{\text{grams}}$$

10) Use the definition of divergence to estimate the flux of $\vec{F}(x,y,z)=\langle 2x+3y,z^2-7y^2,z-4x^2\rangle$ out of the cube centered at (0, 0, 0) with side length 0.01 centimeters. (4 points).

$$\nabla\cdot\vec{F}(0,0,0)\approx\frac{\displaystyle\oiint_{\text{Cube}}\vec{F}\cdot d\vec{S}}{(.01)^3}\Rightarrow\oiint_{\text{Cube}}\vec{F}\cdot d\vec{S}\approx 10^{-6}\,\nabla\cdot\vec{F}(0,0,0).$$

$$\nabla\cdot\vec{F}(0,0,0)=(2-14y+1)\Big|_{(0,0,0)}=3.$$

$$\boxed{\therefore\oiint_{\text{cube}}\vec{F}\cdot d\vec{S}\approx 3(10)^{-6}\text{ flux units}}$$

11) Use the divergence theorem to find $I = \iint_S 3x^2 + 2y^2 + z\, dS$ if S is the sphere $x^2 + y^2 + z^2 = 9$. oriented out. (4 points)

$$I = \iint_S \langle 3x, 2y, 1\rangle \cdot \frac{\langle x, y, z\rangle}{3} \cdot 3\, dS = 3 \iint_S \langle 3x, 2y, 1\rangle \cdot d\vec{S}.$$

$$\underset{\text{D.T.}}{=} 3 \iiint_{\rho \le 3} 3 + 2 + 0\, dV = 15(\text{vol } \rho \le 3) = 15 \cdot \frac{4}{3}\pi (3)^3$$

$$\Rightarrow I = 20\pi(27) = 540\pi$$

12) Use the definition of curl to estimate the work done by $\vec{F}(x, y, z) = \langle 2x + 3y, z^2 - 7y^2, z - 4x^2 \rangle$ on a particle that moves once around the curve C parameterized by $\vec{r}(t) = 0.01\langle \cos(t), \sin(t), 0\rangle$. (2 points)

W = Work.

$$\nabla \times \vec{F}(0,0,0) \cdot \hat{k} \approx \frac{W}{\pi(.01)^2}.$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & z^2-7y^2 & z-4x^2 \end{vmatrix} = \langle 0 - 2z, 0 + 8x, 0 - 3\rangle,$$

So $\nabla \times \vec{F}(0,0,0) = \langle 0, 0, -3\rangle$. $\Rightarrow \nabla \times \vec{F}(0,0,0) \cdot \hat{k} = -3$.

$$\therefore -3 \approx \frac{W}{\pi(.01)^2} \Rightarrow \boxed{W \approx -3\pi(10)^{-4} \text{ work units}}$$