Test 3 100 points Math 200 Name: ______________________

1. (9 points) Use a surface integral to find the mass of the sphere $\rho = 1$ if the density is $\delta(x, y, z) = x^2 + y^2$ grams/cm^2.

$$\text{Mass} = \iint_{\rho=1} x^2+y^2 \, dS \quad ; \quad x^2+y^2 = r^2 = \rho^2 \sin^2\phi = \sin^2\phi \text{ if } \rho=1,$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi \cdot 1^2 \sin\phi \, d\phi \, d\theta$$

$$= 2\pi \cdot \int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi \quad ; \quad u = \cos\phi \Rightarrow du = -\sin\phi \, d\phi$$

$$= 2\pi \cdot \int_1^{-1} (1-u^2)(-1)\, du = 4\pi \int_0^1 (1-u^2)\, du$$

since $1-u^2$ is even and switching limits multiplies by (-1).

$$= 4\pi\left(u - \frac{u^3}{3}\Big|_0^1\right) = \boxed{\frac{8\pi}{3} \text{ grams}}$$

2. (9 points) Use a surface integral to find the flux of $\vec{F}(x, y, z) = \langle x, y, y\rangle$ through the graph of $z = 1 - x^2 + y$ oriented up if $-1 \le x \le 1$ and $-1 \le y \le 1$.

$$d\vec{S} = \pm\langle 2x, -1, 1\rangle \, dx\,dy \; ; \; \text{oriented up} \Rightarrow d\vec{S} = \langle 2x, -1, 1\rangle \, dx\,dy.$$

$$\Rightarrow \Phi = \iint_S \vec{F}\cdot d\vec{S} = \int_{-1}^1\int_{-1}^1 \langle x, y, y\rangle \cdot \langle 2x, -1, 1\rangle \, dx\,dy$$

$$\Rightarrow \Phi = \int_{-1}^1\int_{-1}^1 2x^2 - y + y \, dx\,dy = 2\cdot 2\int_0^1 2x^2\,dx$$

$$\Rightarrow \Phi = 8 \cdot \frac{x^3}{3}\Big|_0^1 = \boxed{\frac{8}{3}}$$

3. (9 points) Find the flux of $\vec{F}(x,y,z) = \langle z, 2, z\rangle$ through the surface parameterized by $\vec{p}(u,v) = \langle uv, u+v, 2v\rangle$ if $u^2+v^2 \le 1$ ~~and~~ oriented with a positive $\hat{\imath}$ - component.

$$\hat{\imath} \quad \hat{\jmath} \quad \hat{k}$$

$$\vec{P}_u\, du = \langle v, 1, 0\rangle\, du$$

$$\times\ \vec{P}_v\, dv = \langle u, 1, 2\rangle\, dv$$

$$d\vec{S} = \pm\langle 2, -2v, v-u\rangle\, du\, dv$$

positive $\hat{\imath}$-component

$$\Rightarrow d\vec{S} = \langle 2, -2v, v-u\rangle\, du\, dv.$$

$$\therefore\ \Phi = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iint_{u^2+v^2\le 1} \langle 2v, 2, 2v\rangle \cdot \langle 2, -2v, v-u\rangle\, du\, dv$$

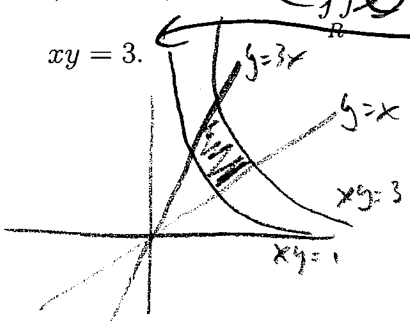
$$\Rightarrow \Phi = \iint_{u^2+v^2\le 1} 2v^2 - 2uv\, du\, dv$$

use polar $\Rightarrow \Phi = \int_0^{2\pi}\int_0^1 (2r^2\sin^2\theta - 2r^2\sin\theta\cos\theta)\, r\, dr\, d\theta$

$$\Rightarrow \Phi = \int_0^{2\pi} [1 - \cancelto{0;\,2\pi}{\cos(2\theta)}] - \cancelto{0;\,2\pi}{\sin(2\theta)}\, d\theta \cdot \int_0^1 r^3\, dr$$

$$\Rightarrow \Phi = 2\pi \cdot \frac{r^4}{4}\Big|_0^1 = \boxed{\frac{\pi}{2}}$$

4. (9 points) Evaluate $I = \iint_R \frac{y}{x}$ ~~y^2~~ $dx\, dy$ using a substitution if R is bounded by $y = x$, $y = 3x$, $xy = 1$, and $xy = 3$. in the first quadrant.



$$u = xy\ ;\ 1 \le u \le 3$$

$$v = \frac{y}{x}\ ;\ 1 \le v \le 3$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = 2v,$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}.$$

$$\therefore\ I = \int_1^3\int_1^3 v\cdot\frac{1}{2v}\, du\, dv$$

$$= \frac{1}{2}\cdot 2\cdot 2 = \boxed{2}$$

8

5. (8 points) Use Lagrange Multipliers to classify the extreme values of $f(x,y) = x + y$ constrained by $x^2 + 4y^2 = \cancel{16}$ 20. Draw a picture of level curves to defend and verify your final answer.

$C(x,y) = x^2 + 4y^2$

$$\begin{cases} \nabla f = \lambda \nabla C \\ x^2 + 4y^2 = 20 \end{cases}$$

$$\Rightarrow \begin{cases} 1 = \lambda \cdot 2x & (1) \\ 1 = \lambda \cdot 8y & (2) \\ x^2 + 4y^2 = 20 & (3) \end{cases}$$

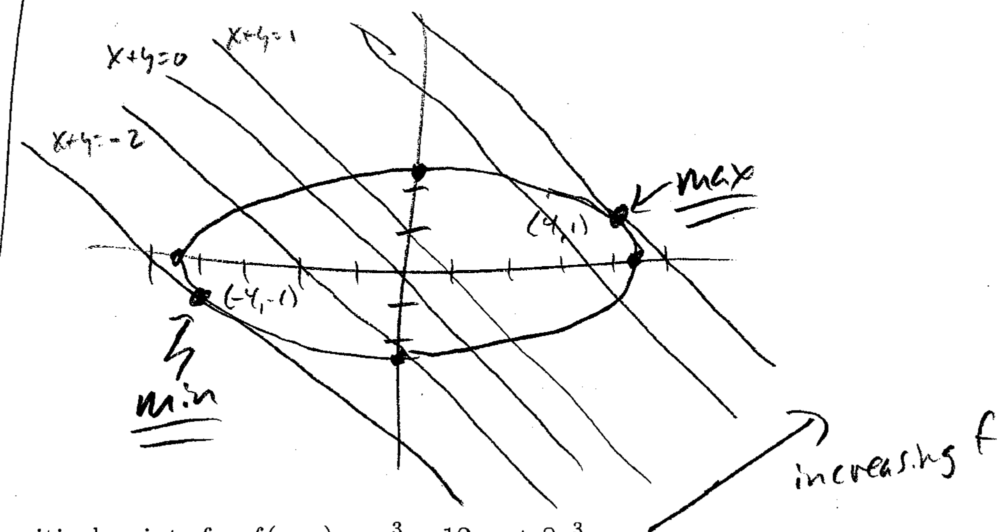
$\frac{(2)}{(1)} \Rightarrow 1 = \frac{4y}{x} \Rightarrow x = 4y.$

$(3) \Rightarrow 16y^2 + 4y^2 = 20$

$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$, and $x = \pm 4$. $(4,1), (-4,-1)$

$f(4,1) = 5$ is a global constrained max

and $f(-4,-1) = -5$ is a global constrained min



6. (8 points) Find and classify all critical points for $f(x,y) = x^3 - 12xy + 8y^3$.

$$\begin{cases} f_x = 3x^2 - 12y = 0 \\ f_y = 24y^2 - 12x = 0 \end{cases}$$

$0 = f_y \Rightarrow 2y^2 = x.$

Subbing into $f_x = 0$

$\Rightarrow 3(2y^2)^2 - 12y = 0$

$\Rightarrow 12y^4 - 12y = 0$

$\Rightarrow y(y^3 - 1) = 0 \Rightarrow y = 0, y = 1$

$\Rightarrow$ c.p.'s are $(0,0)$ and $(2,1)$

$f_{xx} = 6x, \; f_{yy} = 48y, \; f_{xy} = -12$

$H(0,0) = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} < 0$

$\Rightarrow$ $(0,0)$ is a saddle

$H(2,1) = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix} > 0, \; 12 > 0$

$\Rightarrow$ $(2,1)$ is a local min

b/c $f(2,1) = 8 - 24 + 8 = -8$,

but $f(-3,0) = -27 < -8$.

x^2z, $-5xz^2$

7. (8 points) Use the Divergence Theorem to find the flux of $\vec{F}(x,y,z) = \langle \cancel{x^2y}, \cancel{3y^2}, \cancel{-2}xz^2\rangle$ out of the unit cube with x, y, and z between 0 and 1. E = Solid unit cube

~~the boundary of the solid~~

$$\Phi = \iint_{\partial(E)} \vec{F}\cdot d\vec{S} \underset{D.T.}{=} \iiint_E 2xz + 8y - 10xz \; dV$$

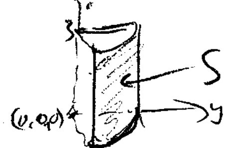
$$\Rightarrow \Phi = \int_0^1\int_0^1\int_0^1 (8y - 8xz)\,dx\,dy\,dz$$

$$= 8\cdot 1\cdot 1\cdot \int_0^1 y\,dy - 8\cdot 1\cdot \int_0^1 x\,dx \cdot \int_0^1 z\,dz$$

$$= 8\cdot\frac{1}{2} - 8\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \boxed{2}$$

8. (8 points) Find the mass of the part of the cylinder $x^2+y^2=4$ for which $x\geq 0$, $y\geq 0$, and $0\leq z\leq 3$ if the density is $\delta(x,y,z) = z + x^2y \quad kg/m^2$.



$$\text{Mass} = \iint_S \delta\, dS$$

$$= \int_0^3\int_0^{\pi/2} (z+x^2y)\cdot 2\,d\theta\,dz$$

$$= \int_0^3\int_0^{\pi/2} 2z + 16\cos^2\theta\sin\theta\,d\theta\,dz$$

$$= \frac{\pi}{2}\cdot\int_0^3 2z\,dz + \int_0^3 -16\cdot\frac{\cos^3\theta}{3}\Big|_0^{\pi/2} dz$$

$$\left(u = \cos\theta \Rightarrow \text{or} \int_0^3\int_1^0 -16u^2\,du\,dz\right)$$

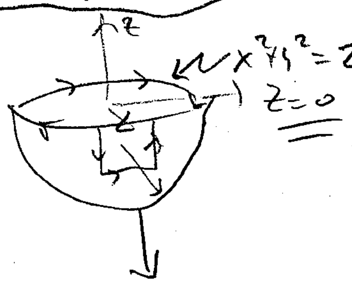
$$\Rightarrow \text{Mass} = \frac{\pi}{2}\cdot\left(z^2\Big|_0^3\right) + 3\cdot\left(\frac{16}{3}\right)$$

$$\Rightarrow \boxed{\text{Mass} = \left(\frac{9\pi}{2} + 16\right)\text{ kg}}$$

9. (8 points) Use Stokes Theorem once to find $I = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = \langle z, x, x^2+y^2 \rangle$ and S is the lower hemisphere $\rho = 2$, $z \le 0$, oriented down.

Bold Bold

$$I \overset{S.T.}{=} \oint_{Bd(S)} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \langle 0, 2\sin(t), 4 \rangle \cdot \langle 2\cos(t), -2\sin(t), 0 \rangle \, dt$$



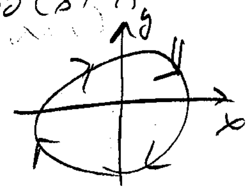
$$\Rightarrow I = \int_0^{2\pi} -4\sin^2(t)\, dt$$

0; 2π

$$I = -2\int_0^{2\pi} 1 - \cancel{\cos(2t)}\, dt$$

$$\Rightarrow \boxed{I = -4\pi}$$

Bd(S') is



$$\Rightarrow \bar{\alpha}(t) = \langle 2\sin(t), 2\cos(t), 0 \rangle$$

$$0 \le t \le 2\pi$$

$$\bar{\alpha}'(t) = \langle 2\cos(t), -2\sin(t), 0 \rangle$$

10. (8 points) Use the Divergence Theorem to find $I = \iint_{Bd(E)} 2x^2 + zy\, dS$ if E is the solid ~~half cylinder~~ Sphere $x^2 + y^2 + z^2 \le$ ~~9~~ 4 ~~with $y \ge 0$~~.

$$I = 2\iint_{Bd(E)} \langle 3x, z, 0 \rangle \cdot \frac{\langle x, y, z \rangle}{2}\, dS$$

(or) $2\iint_{Bd(E)} \langle 3x, 0, y \rangle \cdot \frac{\langle x, y, z \rangle}{2}\, dS$

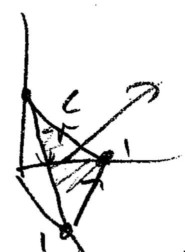
$$\overset{D.T}{=} 2\iiint_E 3 + 0 + 0 \, dV$$

$$= 6 \cdot \frac{4}{3}\pi \cdot 2^3$$

$$= \boxed{64\pi}$$

$\cancel{\langle y, z, \rangle}$? $\cancel{\langle -y^2, -z^2, -x^2 \rangle}$ $\langle -y^2, -z^2, -x^2 \rangle$

11. (8 points) Use Stokes Theorem to find the work done by $\vec{F}(x,y,z) = \langle \cancel{x + y^2, y + z^2, z} - 1 \rangle$ if C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ that is oriented counterclockwise from above.



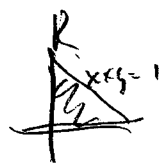
$$\text{Work} = \oint_C \vec{F} \cdot d\vec{s} \underset{S.T.}{=} \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

S = Solid triangle is in the plane $x + y + z = 1 \Rightarrow z = 1 - x - y$ oriented up

$d\vec{S} = \langle 1, 1, 1 \rangle \, dx\,dy$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & -z^2 & -x^2 \end{vmatrix} = \langle 2z, 2x, 2y \rangle = \nabla \times \vec{F}$$

$$\Rightarrow \text{Work} = \iint_R \langle 2z, 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dx\,dy$$



$$= \int_0^1 \int_0^{1-x} 2(1-x-y) + 2x + 2y \, dx\,dy$$

$$= \iint_R 2 \, dx\,dy = 2 \cdot \text{Area}(R)$$

$$= 2 \cdot \tfrac{1}{2} \cdot 1 = \boxed{1}$$

12. (8 points) Prove that the volume of a solid E is equal to $\frac{1}{3} \iint_{Bd(E)} \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

$$\frac{1}{3} \iint_{Bd(E)} \vec{F} \cdot d\vec{S} \underset{D.T.}{=} \frac{1}{3} \iiint_E \nabla \cdot \vec{F} \, dV$$

$$= \frac{1}{3} \iiint_E 3 \, dV$$

$$= \iiint_E 1 \, dV$$

$$= \text{Vol}(E) \; \blacksquare$$