MATH 200 TEST # 3 ANSWERS

(1) A) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ does not exist. $x=0 \Rightarrow \lim_{y\to 0}\frac{2y^2}{y^2}=2$, but

$y=0 \Rightarrow \lim_{x\to 0}\frac{x^2}{x^2}=1$.

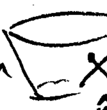
b) $\lim_{(x,y)\to(1,\pi)} \frac{x^2\cos(y)}{x^2+2\sin(y)} = \frac{-1}{1} = \boxed{-1}$ b/c the function is continuous at $(1,\pi)$.

(2) $u = x+y$, $v=\frac{y}{x}$ $\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix} = \frac{1}{x}+\frac{y}{x^2} = \frac{x+y}{x^2} = \frac{u}{x^2} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2}{u}$.

oops. Mass should have been $\frac{1}{x^2}$.

MASS $= \int_1^3\int_1^4 x^2\cdot\frac{x^2}{u}\,dv\,du$. $\frac{u}{x} = 1+\frac{y}{x} \Rightarrow x = \frac{u}{1+v} \Rightarrow$ Mass $= \int_1^3\int_1^4 \frac{u^3}{(1+v)^4}\,dv\,du$

$= \frac{u^4}{4}\Big|_1^3 \cdot \frac{-1}{3}(1+v)^{-3}\Big|_1^4 = \frac{20}{3}(2^{-3}-5^{-3}) = \boxed{39/50}$

(3) 
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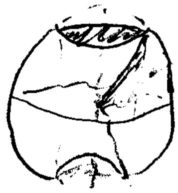
$\hat n$ has negative $\hat k$ component. $\hat n$

$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow$ $\langle\cos\theta, \sin\theta, 1\rangle$ $\times$ $\langle -r\sin\theta, r\cos\theta, 0\rangle$ $\Rightarrow \vec{dS} = -r\langle\cos\theta,\sin\theta,1\rangle\,dr\,d\theta$
$z = r$

$\pm\langle -r\cos\theta, -r\sin\theta, r\rangle$

So $\Phi = -\int_0^{2\pi}\int_2^3 \langle r\sin\theta, -r\cos\theta, r\rangle\cdot r\langle\cos\theta,\sin\theta,1\rangle\,dr\,d\theta$

$= -\int_0^{2\pi}\int_2^3 r^2\,dr\,d\theta = -\frac{2\pi}{3}(27-8) = \boxed{\frac{-38\pi}{3}}$

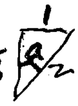
(4) 


$\rho\, d\phi$, $r\,d\theta$

$\rho = 2$

$\Rightarrow dS = \rho^2\sin\phi\,d\phi\,d\theta = 4\sin\phi\,d\phi\,d\theta$, $0\le\theta\le 2\pi$, $\frac{\pi}{6}<\phi<\frac{5\pi}{6}$

MASS $= \int_0^{2\pi}\int_{\pi/6}^{5\pi/6} (10 + \underbrace{2\sin\phi\cos\theta}_{0\text{ Full period}})\,4\sin\phi\,d\phi\,d\theta$


$\sqrt{3}$, 1, a, 2 $\Rightarrow \cos a = \frac{\sqrt{3}}{2} \Rightarrow a = \frac{\pi}{6}$

$= 80\pi\left(-\cos\phi\Big|_{\pi/6}^{5\pi/6}\right) = 80\pi\left(\frac{\sqrt3}{2}+\frac{\sqrt3}{2}\right) = \boxed{80\pi\sqrt{3}}$ grams.

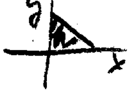
(5) 
The triangle $=\partial R$

Where R is the triangular region in $x+y+z=1$; use $\underline{z=1-x-y}$.

$\therefore$ Stokes' Theorem $\Rightarrow$ Work $= \iint_R \nabla\times\vec F\cdot\vec{dS}$.

$\nabla\times\langle z,y,z\rangle = \langle 0,1,0\rangle$, and $\vec{dS} = \langle 1,1,1\rangle\,dx\,dy$,

Work $= \iint 1\,dx\,dy$ = area of triangle $= \boxed{\frac{1}{2}}$



(6) $\text{Flux} = \underset{\partial Box}{\oiint} \vec{F}\cdot d\vec{S} \underset{\text{Div thm}}{=} \underset{Box}{\iiint} \overset{0 \text{ by symmetry}}{\underset{\text{odd}}{\cancel{2x}}} + 3 + 1 \, dV = 4 \cdot \text{vol}(\text{Box})$

$= 4\cdot 2\cdot 4\cdot 2 = \boxed{64}$

(7) A) $f_x = 3x^2 - 3x = 0 \Rightarrow x = 0, 1$
$f_y = 3y^2 - 3y = 0 \Rightarrow y = 0, 1$

$\boxed{\text{So the critical points are } (0,0), (1,0), (0,1), (1,1)}$

B) $H = \begin{bmatrix} 6x-3 & 0 \\ 0 & 6y-3 \end{bmatrix} \Rightarrow$ $H(0,0) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow \boxed{f(0,0) = 0 \text{ is a local max}}$

$H(1,0) = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow \boxed{(1,0) \text{ is at a saddle}}$

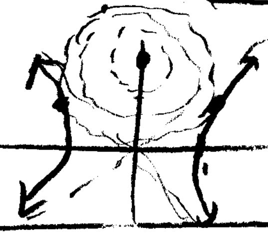
Similarly, $\boxed{(0,1) \text{ is at a saddle}}$; $H = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$ $H(1,1) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \boxed{f(1,1) = -1 \text{ is local min}}$

(8) $\nabla f = \lambda \nabla c \Rightarrow 2x = 2\lambda x \Rightarrow \cancel{x=0}$ or $\lambda = 1$

$\boxed{f(x,y) = x^2 + (y-2)^2}$

$2(y-2) = -2\lambda y$ $\Downarrow$ $\Downarrow$
$x^2 - y^2 = 1$ no solution $2y - 4 = -2y \Rightarrow y = 1$
$\Rightarrow x^2 = 2.$

So the critical points are $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$.
$f(\pm\sqrt{2}, 1) = 2 + 1 = 3$, and I claim $\boxed{3}$ is a minimum by considering the picture:



(9) $\underset{E}{\iiint} (x^2+y^2+z^2)^{-1} dV = \int_0^{2\pi}\int_2^4\int_0^{\pi/6} \sin\phi \, d\phi \, d\rho \, d\theta = 2\pi \cdot 2 \cdot \left(-\cos\phi \Big|_0^{\pi/6}\right)$

$= \boxed{4\pi\left(1 - \frac{\sqrt{3}}{2}\right)}$

(10) $\underset{E}{\iiint} y^2 \, dV = \int_0^{\pi/2}\int_0^1\int_0^1 r^2 \sin^2\theta \, r \, dr \, dz \, d\theta$

$= 1 \cdot \left(\frac{r^4}{4}\Big|_0^1\right) \cdot \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta = \frac{1}{8}\left(\frac{\pi}{2} - \left(\frac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right) = \boxed{\frac{\pi}{16}}\right.$

(11) $\nabla \cdot \vec{F}(P) \approx \dfrac{\text{Flux}}{\text{Vol}} \Rightarrow \text{Vol} \approx \dfrac{\text{Flux}}{\nabla \cdot \vec{F}(P)}$, so

$$\text{Vol} \approx \frac{10}{5} = \boxed{2}$$

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(12) $\dfrac{\partial g}{\partial n} = D_{\hat{n}} g = \nabla g \cdot \hat{n}$, and $\nabla^2 g = \nabla \cdot \nabla g$. So

$$\oiint_{\partial D} f \frac{\partial g}{\partial n}\, dS = \oiint_{\partial D} (f \nabla g) \cdot \hat{n}\, dS \underset{\text{Div thm}}{=} \iiint_D \nabla \cdot (f \nabla g)\, dV$$

$$= \iiint_D \nabla f \cdot \nabla g + f(\nabla \cdot \nabla g)\, dV$$

$$= \iiint_D \nabla f \cdot \nabla g\, dV + \iint_D f \nabla^2 g\, dV \quad \blacksquare$$