Final 100 points Math 210 Name: ______________________________

Always show work or defense for full credit.

1. Write "True" or "False" with reason or counterexample.

   (a) (6 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ so that $L(x, y) = (x^2 - y, y^2)$ is a linear transformation.

   (b) (6 points) If A is a 4 x 4 matrix with eigenvalue λ, then an eigenvalue of $3A^2$ is $3\lambda^2$.

   (c) (6 points) If A is an m x n matrix and $\vec{x}$ is in $N(A^T A)$, then $\vec{x}$ is in $N(A)$.

2. $K = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$ for both parts of this question.

(a) (4 points) Notice that $K - I$ is singular. Find the eigenvalues of K.

(b) (4 points) Let p be a real number. For what p is $K + pI$ a positive definite matrix?

3. (8 points) Find the projection of the vector $(0, 20, 20, 20)$ onto the space spanned by the vectors $(1, 1, 1, 1)$ and $(-2, -1, 1, 2)$.

4. A is a 2 x 2 matrix with real entries, eigenvalues 0.2 and 1, $N(A - 0.2I) = c(1, 4)$, and $N(A - I) = c(2, 9)$ for all parts of this problem.

   (a) (4 points) Is A symmetric?

   (b) (4 points) Write the solution to the system of differential equations $\vec{u}^{\,'}(t) = A\vec{u}(t)$ if $\vec{u}(0) = (1, 5)$.

   (c) (4 points) What is $\lim_{n\to\infty} A^n$?

5. (4 points) If $M = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$ is a Markov matrix modeling a system with 1300 objects, what is the steady state vector?

6. (6 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ so that $L(x, y) = (2x + 3y, 3x + 2y)$ is a linear transformation, and $B = (2, 1), (1, 1)$ is a basis for $\mathbb{R}^2$. Find $[L]_B$.

7. (8 points) Find the general solution to $A\vec{x} = \vec{b}$ if $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\vec{b} = (-8, 2, 4)$.

8. (6 points) Find the matrix of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates 90° counterclockwise about the z-axis and then reflects across the xy-plane.

9. (8 points) Find the QR-factorization of $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$.

10. (8 points) Find the $X\Lambda X^{-1}$ factorization of $K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The Spectral Theorem guarantees X^{-1} is easy to calculate.

11. $U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, and $A = U\Sigma V^T$ is the SVD of A for this problem.

(a) (2 points) What is the dimension of $N(A)$? ________

(b) (2 points) What is a basis for $C(A)$? ____________________________________

(c) (2 points) $C(A^T) =$ ____

12. (8 points) Let $B = \begin{bmatrix} 3 & 3 \\ 4 & 4 \\ 0 & 0 \end{bmatrix}$. Solve $B\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\vec{x}^+$.