Practice Final Exam Problems From MIT

1) (a) - Find the eigenvalues and eigenvectors of $A=\begin{bmatrix}3&1&4\\0&1&5\\0&1&5\end{bmatrix}$.

(b) - Write the vector v = (1, 1, 1) as a linear combination of eigenvectors of A.
- Find the vector $A^{10}v$

(c) If you solve $\frac{d\mathbf{u}}{dt}=-A\mathbf{u}$ (notice the minus sign), with **u**(0) a given vector, then as t approaches ∞ the solution **u**(t) will always approach a multiple of a certain vector w.
- Find this steady-state vector w.

2) Suppose A has rank 1, and B has rank 2 (A and B are both 3 x 3 matrices).
(a) - What are the possible ranks of A + B?
(b) - Give an example of each possibility you had in (a).
(c) - What are the possible ranks of AB?
- Give an example of each possibility.

3) (a) - Find the three pivots and the determinant of $A=\begin{bmatrix}1&0&-1\\0&1&1\\-1&1&0\end{bmatrix}$.

(b) – Find the eigenvalues of A.
(c) There are orthonormal eigenvectors $\vec{x}_1,\vec{x}_2,\vec{x}_3$.
- Prove that: $A=\lambda_1\vec{x}_1\vec{x}_1^T+\lambda_2\vec{x}_2\vec{x}_2^T+\lambda_3\vec{x}_3\vec{x}_3^T$
(d) Find the QR factorization of A.
(e) The orthonormal vectors q1; q2; q3 from the QR factorization of A form a basis B for $\mathbb{R}^3$. What are the B – coordinates of the third column of A?

4) This problem is about x + 2y + 2z = 0, which is the equation of a plane through 0 in $\mathbb{R}^3$.
(a) - That plane is the nullspace of what matrix A?
- Find an orthonormal basis for that nullspace (that plane).
(b) That plane is the column space of many matrices B.
- Give two examples of B.
(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)
- What is the rank of P?

5) Suppose **v** is any unit vector in $\mathbb{R}^3$. This question is about the matrix $H=I-2\mathbf{v}\mathbf{v}^T$.
(a) - Multiply H times H to show that $H^2=I$.
(b) - Show that H passes the tests for being a symmetric matrix and an orthogonal matrix.
(c) - What are the eigenvalues of H?

6) (a) - Find the closest straight line y = Ct + D to the 5 points:
(t; y) = (- 2; 0); (- 1; 0); (0; 1); (1; 1); (2; 1):
(b) - The word "closest" means that you minimized which quantity to find your line?
(c) - If A^TA is invertible, what do you know about its eigenvalues and eigenvectors?

7) This symmetric Hadamard matrix $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ has orthogonal columns and $H^2 = 4I$.

(a) What are the eigenvalues of H?
(b) What is the determinant of H?
(c) What are the singular values of H?

8) TRUE or FALSE?
(a) Suppose you have 101 vectors v1; v2; : : : ; v101 in $\mathbb{R}^{100}$.
- Each vi is a combination of the other 100 vectors.
- Three of the vi's are in the same 2-dimensional plane.

(b) Suppose a matrix A has repeated eigenvalues 7; 7; 7, so $|A - \lambda I| = (7-\lambda)^3$.
- Then A certainly cannot be diagonalized.
- The Jordan form of A must be $J = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$

(c) Suppose A and B are 3 x 5.
- Then rank(A + B) ≤ rank(A) + rank(B)

(d) Suppose A and B are 4 x 4.
- Then det(A + B) ≤ det(A) + det(B)

(e) Suppose **u** and **v** are orthonormal, and call the vector **b** = 3**u** + **v**. Take V to be the line of all multiples of **u** + **v**.
- The projection of **b** onto V is 2**u** + 2**v**.

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$ for a fixed function f. The input is x, the output is T(x).
- Then T is always a linear transformation.

9) If $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$ and **b** equals the sum of the columns of A, what is the complete solution for $A\mathbf{x} = \mathbf{b}$?

10) (a) Find the rank – 1 decomposition of the SVD for $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. (b) Find x^+ if $\mathbf{b} = \mathbf{1}$.

(c) Find all $\hat{\mathbf{x}}$ so that $A^T A\hat{\mathbf{x}} = A^T \mathbf{1}$.

Answers (First Edition – be aware of typing errors and just plain errors and please let me know if you find any.)

1A) $\lambda = 0,3,6$; N(A) = c(1,-15,3), N(A-3I) = c(1,0,0), and N(A-6I) = c(5,3,3)

1B) $6^{10}(5/3,1,1)-(2\cdot 3^9,0,0)$.

1C) $\vec{u}(t) = c_1e^{-3t}(1,0,0)+c_2e^{-6t}(5,3,3)+\mathrm{c}_3(1,-15,3)$; $\vec{u}(t)\to c_3\left(1,-15,3\right)$ as $t\to\infty$.

2AB) rank 1: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}$; rank 2: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$; rank 3: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

2C) rank 0: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$; rank 1: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

3A) det(A) = -2; pivots = 1, 1, -2.

3B) $\lambda = -1,1,2$

3C) The spectral theorem implies $A = Q\Lambda Q^T$. The rank – 1 decomposition of this is $A = \lambda_1\vec{x}_1\vec{x}_1^T+\lambda_2\vec{x}_2\vec{x}_2^T+\lambda_3\vec{x}_3\vec{x}_3^T$.

3D) $A = QR = \begin{bmatrix}1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{3}\\0&2/\sqrt{6}&-1/\sqrt{3}\\-1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{3}\end{bmatrix}\begin{bmatrix}2/\sqrt{2}&-1/\sqrt{2}&-1/\sqrt{2}\\0&3/\sqrt{6}&1/\sqrt{6}\\0&0&-2/\sqrt{3}\end{bmatrix}$

3E) $\left(-1/\sqrt{2},1/\sqrt{6},-2/\sqrt{3}\right)_B$.

4A) A = [1 2 2]. $(2,-1,0)/\sqrt{5}$, $(2,4,-5)/3\sqrt{5}$

4B) $B = \begin{bmatrix}2&2\\-1&4\\0&-5\end{bmatrix}$ or $B = \begin{bmatrix}4&2\\-2&4\\0&-5\end{bmatrix}$ or $B = \begin{bmatrix}2&2&0\\-1&4&0\\0&-5&0\end{bmatrix}$ or ….

4C) $P = \begin{bmatrix}2&2\\-1&4\\0&-5\end{bmatrix}\begin{bmatrix}5&0\\0&45\end{bmatrix}^{-1}\begin{bmatrix}2&-1&0\\2&4&-5\end{bmatrix}$. Rank P = 2.

5A) Hint: $\mathbf{v}^T\mathbf{v}$ = 1.

5B) Show $H^T = H$; then $H^TH = H^2 = I$ from 5A.

5C) $\lambda = 1,1,-1$

6A) $\hat{y} = .3t+.6$

6B) $\|\mathbf{e}\|^2 = \|\mathbf{b}-projection\|^2 = \sum_{i=1}^{5}(b_i-\hat{y}_i)^2$.

6C) positive eigenvalues and there is an orthonormal basis of eigenvectors.

7A) 2, 2, -2, -2 7B) 16 7C) 2, 2, 2, 2

8A) False, False
8B) False, False
8C) True. The rows of A + B are contained in V = span of (the rows of A and the rows of B), so rank of A + B is less than or equal to the dimension of V. But A and B may share dependent rows, so dim V is less than or equal to rank A + rank B. Thus rank (A + B) is less than or equal to rank A + rank B.
8D) False
8E) True – use coordinates.
8F) False.

9) $(1,1,1,1)+c_1(-1,0,-2,1)+c_2(-2,1,0,0)$

10A) $\sqrt{70}\,\frac{\begin{bmatrix}1\\2\\3\end{bmatrix}}{\sqrt{14}}\frac{\begin{bmatrix}1 & 2\end{bmatrix}}{\sqrt{5}}$

10B) $\frac{3}{35}\begin{bmatrix}1\\2\end{bmatrix}$

10C) $\frac{3}{35}\begin{bmatrix}1\\2\end{bmatrix}+c\begin{bmatrix}2\\-1\end{bmatrix}$