1. Find a basis for $N(R)$ and $N(R^T)$ if $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find the complete solution for the following systems. Use vectors to write that solution.

2. $\begin{cases} x + 3y + 3z = 1 \\ 2x + 6y + 9z = 5 \\ -x - 3y + 3z = 5 \end{cases}$

3. $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

4. $x + y + z = 4$

5. $A\vec{x} = \vec{b}$ if $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

6. Factor $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix}$ into LU form, find a basis for the four fundamental spaces of A, and find the rank of A. Your answers should be in terms of q.

7. Show by elimination that (b_1, b_2, b_3) is in the column space of $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}$ if $b_3 - 2b_2 + 4b_1 = 0$.

   What linear combination of rows of A gives the zero row?

8. If A is 7 x 9 with rank 5, what are the dimensions of the four fundamental spaces of A? Defend your answer.

Find a basis for each of these subspaces of $\mathbb{R}^4$.

9. All vectors with components that sum to zero.

10. All vectors perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

Which of the following are bases for $\mathbb{R}^3$? Defend your answers.

11. $(1, 1, -1)$, $(2, 3, 4)$, $(4, 1, -1)$, $(0, 1, -1)$

12. $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 0)$

13. $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 6)$.

14. Without elimination find the dimensions and bases for the four fundamental spaces of $B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$.

15. A is an m x n matrix with rank r and there is a $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution. Why must $A^T\vec{y} = \vec{0}$ have nonzero solutions?

Construct a matrix with the required property or explain why it is impossible.

16. $C(A)$ contains $(1, 1, 0)$ and $(0, 0, 1)$, and $C(A^T)$ contains $(1, 2)$ and $(2, 5)$.

17. $C(B)$ has basis $(1, 1, 3)$ and $N(B)$ has basis $(3, 1, 1)$.

18. $\dim N(D) = 1 + \dim N(D^T)$.

Answer "True" or "False" and then defend your answer.

19. The vectors $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 2, 1)$ are independent.

20. The vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$ are independent.

21. The columns of a matrix A are a basis for the column space of A.

22. If A is n x n then dim $N(A) =$ dim $N(A^T)$.

23. CAS Problem (3 points): Use a CAS for the following. Submit a printed copy of your commands and answers. K is the 9 x 9 second difference matrix with $K_{ii} = 2$ on the main diagonal and -1's on the diagonal above and below the main diagonal. Solve $K\vec{x} = \vec{b}$ if $\vec{b} = (10, 10, \ldots, 10)$. Plot the nine points (i, x_i) for $i = 1, 2, \ldots, 9$. This second difference matrix K is a discrete negative second derivative. Why should the plot be a parabola?

Brief answers

1. $N(R)$: $(-2,-4,1,0)$, $(-3,-5,0,1)$; $N(R^T)$: $(0,0,1)$

2. $\vec{x} = c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

3. $\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$

4. $\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

5. No solution.

6. $q \neq 2 \implies$ rank $A = 3$, $C(A)$ and $C(A^T)$ have same basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; and $N(A)$ and $N(A^T)$ equal $\vec{0}$, so they don't have a basis. $q = 2 \implies$ rank $A = 2$; $C(A)$: $(1,1,1)$, $(0,1,1)$; $C(A^T)$: $(1,0,1)$, $(0,1,1)$; $N(A)$: $(1,1,-1)$; $N(A^T)$: $(0,1,-1)$.

7. Row3 $-$ 2 Row2 $+$ 4 Row1 $= 0$.

8. dim $C(A) =$ dim $C(A^T) = 5$, dim $N(A) = 4$, and dim $N(A^T) = 2$.

9. $(1,-1,0,0)$, $(1,0,-1,0)$, $(1,0,0,-1)$

10. $(-1,1,1,0)$, $(-1,1,0,1)$

11. No

12. Yes

13. No

14. dim $C(B) =$ dim $C(B^T) = 1$, dim $N(B) = 0$, and dim $N(B^T) = 2$. The bases are: $C(B)$: (1, 4, 5) $C(B^T)$: [1] $N(B)$: none; $N(B^T)$: $(4,-1,0)$, $(5,0,-1)$

15. Hint: use dim $C(A)$ + dim $N(A^T)$ = m.

16. Possible

17. Impossible

18. Possible (Try D 1 x 2)

19. True

20. False

21. False

22. True