1. With b = 0, 8, 8, 20 at t = 0, 1, 3, 4, set up and solve the normal equations $A^T A\hat{x} = A^T\vec{b}$ to find the equation for the best straight line that fits the data. Find the components of the projection $\vec{p}$ of $\vec{b}$ onto $C(A)$ and the components of the error vector $\vec{e}$ and draw them on a graph that includes the line. The best straight line minimizes the magnitude of $\vec{e}$. $\vec{e}$ is perpendicular to $\vec{p}$.

2. Find the closest line $b = Dt$ through the origin to the four points b = 0, 8, 8, 20 at t = 0, 1, 3, 4. Without actually calculating the new error vector, decide if the magnitude of this error vector is larger or smaller than the one from problem 1. Defend your answer.

3. Find the closest parabola $b = C + Dt + Et^2$ to the four points $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$. Even though the answer is nice, I suggest you use a CAS; but set up the normal equations before using your technology.

4. Find the best line $b = C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at $t = -2, -1, 0, 1, 2$.

5. Find orthonormal vectors $\vec{q}_1$ and $\vec{q}_2$ in the plane spanned by $\vec{a} = (1, 3, 4, 5, 7)$ and $\vec{b} = (-6, 6, 8, 0, 8)$.

6. Find orthonormal vectors $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$ such that $\vec{q}_1$ and $\vec{q}_2$ span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

7. In the problem 6, which of the four fundamental subspaces of A contains $\vec{q}_3$?

8. For A in problem 6, find the projection of $(1, 2, 7)$ onto $C(A)$.

9. (a) Find orthogonal vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ by Gram-Schmidt from $\vec{a} = (1, 1, 2)$, $\vec{b} = (1, -1, 0)$, and $\vec{c} = (1, 0, 4)$.
   (b) Find the QR factorization for the matrix with columns $\vec{a} = (1, 1, 2)$, $\vec{b} = (1, -1, 0)$, and $\vec{c} = (1, 0, 4)$.

10. (a) Find a basis for the subspace S in $\mathbb{R}^4$ equal to all solutions of $x_1 + x_2 + x_3 - x_4 = 0$.
    (b) Find a basis for the orthogonal complement $S^\perp$.
    (c) Find $\vec{b}_1$ in S and $\vec{b}_2$ in $S^\perp$ so that $\vec{b}_1 + \vec{b}_2 = \vec{b} = (1, 1, 1, 1)$.

11. Find the inverse of $W = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$ quickly without using elimination or a calculator.

12. (a) Choose c so that $Q = c\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ is an orthogonal matrix.
    (b) Project $\vec{b} = (1, 1, 1, 1)$ onto the first column.
    (c) Project $\vec{b} = (1, 1, 1, 1)$ onto the space spanned by the first and second column.

Answer "True" or "False" and then defend your answer. Give examples in both cases.

13. Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

14. If the 3 x 2 matrix Q has orthonormal columns, then $\|Q\vec{x}\|$ always equals $\|x\|$.

15. Use a substitution to show that $\displaystyle\int_0^{2\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(u)\,du$ whenever $f(x)$ is a 2π-periodic function.

For the remainder of the homework, the inner product $\langle f, g\rangle = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cdot g(x)\,dx$.

16. Simplify $\langle f, g\rangle$ if (a) $f(x)\cdot g(x)$ is odd; (b)$f(x)\cdot g(x)$ is even.

17. Use a trigonometric identity to show that $\cos(jx)$ and $\cos(kx)$ are orthogonal when j and k are different integers. What is the inner product when $j = k$?

Compute the Fourier series for the 2π - periodic function defined in the next two problems.

18. $f(x) = 0$ from $-\pi$ to 0 and $f(x) = 1$ from 0 to π.

19. $g(x) = x$ from 0 to 2π.

20. The 2π - periodic function $f(x) = -1$ from $-\pi$ to 0 and $f(x) = 1$ from 0 to π has Fourier series $\dfrac{4}{\pi}\left(\sin(x) + \dfrac{\sin(3x)}{3} + \dfrac{\sin(5x)}{5} + \cdots\right)$. Use a CAS to graph $f(x)$ (with $f(0) = 0$) and the first 10 terms of the Fourier series of $f(x)$ on the same coordinate axes. The Gibbs phenomenon is the oscillation that overshoots the jump discontinuity - it never dies down with more terms.

21. Let a 2π - periodic function be defined by $g(x) = x$ from $-\pi$ to π. Find the following.
(a) The projection of $g(x)$ onto $\sin(x)$.
(b) The "angle" between $g(x)$ and $\sin(x)$.

22. Answer "True" or "False" and then defend your answer.
If f(x) is an even 2π - periodic function, then its Fourier series does not have any sine terms.

23. CAS Problem (3 points): Use a CAS for the following. Submit a printed copy of your commands and answers.

Find the QR-factorization for $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Then rescale each column of Q (separately) so that the entries are integers.

In Matlab, A=eye(4)-diag([1 1 1],$-$1).

Not graded: conjecture what the rescaled columns will be for a similar A that has size n x n.

Brief answers

1. $b = 1 + 4t$; $\vec{p} = (1,\ 5,\ 13,\ 17)$; $\vec{e} = (-1, 3, -5, 3)$.

2. $b = \dfrac{56}{13}t$; error is larger than error in problem 1.

3. $b = 2 + \dfrac{4}{3}t + \dfrac{2}{3}t^2$

4. $b = 1 - t$.

5. $(0.1,\ 0.3,\ 0.4,\ 0.5,\ 0.7)$ and $(-0.7, 0.3, 0.4, -0.5, 0.1)$

6. $\vec{q}_1 = \dfrac{(1,2,-2)}{3}$, $\vec{q}_2 = \dfrac{(2,1,2)}{3}$, and $\vec{q}_3 = \dfrac{(-2,2,1)}{3}$

7. Left Null space.

8. (3,0,6)

9. (a) $\vec{A} = (1,\ 1,\ 2)$, $\vec{B} = (1, -1, 0)$, $\vec{C} = (1, 1, -1)$

   (b) $QR = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 & 3\sqrt{3}/\sqrt{2} \\ 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -\sqrt{3} \end{bmatrix}$

10. (a) $(1,\ 0,\ 0,\ 1)$, $(1, 0, -1, 0)$, $(1, -1, 0, 0)$ (b) $(1, 1, 1, -1)$

    (c) $\vec{b}_2 = (0.5, 0.5, 0.5, -0.5)$ and $\vec{b}_1 = \vec{b} - \vec{b}_2 = (0.5, 0.5, 0.5, 1.5)$

11. $W^{-1} = W^T$

12. (a) $c = 0.5$ (b) $(-0.5, 0.5, 0.5, 0.5)$ (c) $(0,\ 0,\ 1,\ 1)$

13. True

14. True

15. Show the integral from π to 2π equals the integral from $-\pi$ to 0 by substituting $u = x - 2\pi$.

16. (a) 0 (b) $\langle f, g \rangle = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \cdot g(x)\, dx$

17. inner products should be zero if $j \neq k$ and one if $j = k$.

18. $\dfrac{1}{2} + \dfrac{2}{\pi} \left(\sin(x) + \dfrac{\sin(3x)}{3} + \dfrac{\sin(5x)}{5} + \cdots \right)$.

19. $\pi - 2 \left(\sin(x) + \dfrac{\sin(2x)}{2} + \dfrac{\sin(3x)}{3} + \cdots \right)$.

20. Show a graph.

21. (a) $2\sin(x)$ (b) $\cos^{-1}\left(\dfrac{\sqrt{6}}{\pi} \right)$

22. True