1. If a 4 x 4 matrix has $\det(A) = 0.5$, what are $\det(2A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$?

2. Use row operations to simplify and compute $\det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$.

3. From $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ find the determinants of A^2, A^{-1}, and $A - \lambda I$. What numbers make $\det(A - \lambda I)$ equal to zero? For these numbers what can be said about $N(A - \lambda I)$?

4. If $\det(A) = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6$, find $\det(B) = \begin{vmatrix} \text{row 1 + row 2 + row 3} \\ \text{row 1 + row 2} \\ \text{row 1} \end{vmatrix}$. Defend your answer.

5. Answer "True" or "False" and then defend your answer. The determinant of $4A$ is $4|A|$.

6. Find the determinant of $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$.

7. Find the cofactor matrix C of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and the product of A and C^T. Use this to find A^{-1} and $|A|$.

8. The cofactor matrix of B is $C = \begin{bmatrix} 1 & -1 & 2 \\ 2 & a & -4 \\ b & 1 & 2 \end{bmatrix}$, and row 1 of B is $\begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$.
   (a) What is the determinant of B? How do you know?
   (b) What must a and b equal and how do you know?
   (c) What is $\left(B^{-1}\right)_{32}$? How do you know?

9. Use Cramers Rule to solve $\begin{matrix} ax + by + cz = 1 \\ dx + ey + fz = 0 \\ gx + hy + kz = 0 \end{matrix}$ for y only. You may use D in your answer to represent the 3 x 3 determinant for the coefficent matrix of the system.

Find the following for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in the next three problems.

10. The eigenvalues and eigenspaces of A.

11. The eigenvalues and eigenspaces of $A + I$ and $A + kI$.

12. The eigenvalues and eigenspaces of A^2, A^{-1}, and A^n for any nonzero integer n.

What do you do with the equation $A\vec{x} = \lambda\vec{x}$ in order to prove the next three statements? Give the proofs.

13. λ^2 is an eigenvalue of A^2.

14. λ^{-1} is an eigenvalue of A^{-1}.

15. $\lambda + 1$ is an eigenvalue of $A + I$.

16. Find the eigenvalues for $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. If P is a permutation matrix, then $P\vec{1} = \vec{1}$.

17. Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$.

Find the eigenvalues and eigenspaces of A, B, and C.

18. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

19. $B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

20. $C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

Answer "True" or "False" and then defend your answer.

21. If B is obtained from A using elementary row operations, then the eigenvalues of A and B are the same.

22. The eigenvalues of A and A^T are the same.

23. CAS Problem (3 points): Use a CAS for the following. Submit a printed copy of the commands and answers. Find the eigenvalues for A=ones(8)-eye(8). Verify that the sum of the eigenvalues equals the trace of A and that the determinant of A equals the product of the eigenvalues. Not graded: can you conjecture what the eigenvalues of a matrix similar to A but of size n x n will be?

Brief answers

1. $\det(2A) = 8$ $\quad\det(-A) = 0.5$ $\quad\det(A^2) = 0.25$ $\quad\det(A^{-1}) = 2$

2. 0

3. 100, $\quad 0.1,$ $\quad \lambda^2 - 7\lambda + 10,$ $\quad \lambda = 2, 5.$
   2 and 5 are eigenvalues of A and the corresponding null spaces are nonzero.

4. -6

5. False

6. $\det B = 48.$

7. $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad \det A = 4$

8. (a) $\det B = 3$ $\quad$ (b) $a = 7$, $b = 8$ $\quad$ (c) $\left(B^{-1}\right)_{32} = \frac{-4}{3}$

9. $y = \frac{fg - dk}{D}$

10. $\lambda = 5$ with $N(A - 5I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda = -1$ with $N(A + I) = c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Notice that the eigenspaces do not change in the next two problems.

11. $A + I$ has $\lambda = 6$ with eigenspace $= c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda = 0$ with eigenspace $= c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. $A + kI$ has $\lambda = 5 + k$ with eigenspace $= c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda = k - 1$ with eigenspace $= c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

12. For A^n, $\lambda = 5^n$ with $N(A^n - I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda = (-1)^n$ with $N(A^n + I) = c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. So A^2 has $\lambda = 5^2$ and A^{-1} has $\lambda = 5^{-1}$.

13. Multiply by A on both sides.

14. Multiply by A^{-1} on both sides.

15. Add $\vec{x}$ on both sides.

16. $\lambda_{P_1} = 1, \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$ $\quad$ and $\quad$ $\lambda_{P_2} = 1, 1, -1.$

17. $\lambda = 0, 0$ with $N(A) = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $\quad$ and $\quad$ $\lambda = 6$ with $N(A - 6I) = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

18. $\lambda_A = 1, 4, 6$, $N(A - I) = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $N(A - 4I) = c\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $N(A - 6I) = c\begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$.

19. $\lambda_B = 2, 1 \pm 3i$, $N(A-2I) = c\begin{bmatrix}0\\1\\0\end{bmatrix}$, $N(A-(1+3i)I) = c\begin{bmatrix}1\\0\\-i\end{bmatrix}$, $N(A-(1-3i)I) = c\begin{bmatrix}1\\0\\i\end{bmatrix}$, $c \in \mathbb{C}$.

20. $\lambda_C = 0, 0, 6$, $N(A) = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $N(A-6I) = c\begin{bmatrix}1\\1\\1\end{bmatrix}$.

21. False

22. True