1. Factor $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ into $X\Lambda X^{-1}$ form.

2. If $A = X\Lambda X^{-1}$ , then what are $A^3$ and $A^{-1}$ in terms of $X$ and $\Lambda$?

3. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is ____ , not ____. Change one entry to make A diagonalizable. Which entries could you change and how? Explain your answer.

4. $A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every eigenvalue of $A$ has magnitude less than ____. . Which of these matrices approaches 0 as $k \to \infty$?

$$A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$$

5. Diagonalize $B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix}$ and compute $X\Lambda^k X^{-1}$ to prove $B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

6. Find the eigenvalues, eigenspaces, and the kth power of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Notice how the eigenspaces are orthogonal; this should help you find the inverse of $X$ quickly when diagonalizing.

7. Show that $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are similar by finding $M$ so that $B = M^{-1}AM$. Verify your answer by multiplying.

8. Find the eigenvalues and steady state eigenvectors for these Markov matrices.

$$C = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0,5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

9. Diagonalize $A = \begin{bmatrix} 0.90 & 0.15 \\ 0.10 & 0.85 \end{bmatrix}$ and then find $\lim_{k\to\infty} A^k$.

10. (a) Define $\vec{u}_n$ and $\vec{u}_0$, and find a transition matrix $A$ so that $\vec{u}_{n+1} = A\vec{u}_n$ represents the following process. A company owns 3720 trucks that are originally distributed equally to Los Angeles, Dallas, and Chicago. Each week 10% of the LA trucks go to Dallas and 20% go to Chicago; 20% of the Dallas trucks go to LA and 30% go to Chicago; and 40% of the Chicago trucks go to LA and 50% go to Dallas.
(b) Find the steady state if it exists, otherwise explain how you know it doesn't.

11. Solve $\dfrac{d\vec{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \vec{u}$ using eigenvalues and eigenvectors if $\vec{u}(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

12. $r(t)$ and $w(t)$ represent the populations of rabbits and wolves respectively. Solve

$$dr/dt = 6r - 2w$$
$$dw/dt = 2r + w$$

if $r(0) = w(0) = 30$. After a long time, what is the ratio of rabbits to wolves?

13. A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to $v - w$, so $dv/dt = w - v$ and $dw/dt = v - w$.
(a) Show that the total $v + w$ is constant.
(b) Find $\vec{u}$, the matrix in $\frac{d\vec{u}}{dt} = A\vec{u}$,the eigenvalues, and the eigenspaces.
(c) Find $v$ and $w$ at $t = 1$ and as $t \to \infty$.

14. Find $e^{At}$ if $\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$.

15. Find an orthogonal matrix $Q$ so that $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} = Q\Lambda Q^T$ .

16. Write $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ in the rank 1 decomposition form of the spectral theorem.

Answer "True" or "False" and then defend your answer.

17. If the columns of $X$ are linearly independent, then $A = X\Lambda X^{-1}$ is invertible.

18. If the columns of X are linearly independent, then X is diagonalizable.

19. If the eigenvalues of $A$ are 2, 2, 5, then A is invertible.

20. If the eigenvalues of $A$ are 2, 2, 5, then A is diagonalizable.

21. If $C$ is similar to $A$ and also similar to $B$, then $A$ and $B$ are similar.

22. The square of a Markov matrix is Markov.

23. CAS Problem (3 points): Use a CAS for the following. Submit a printed copy of the commands and answers. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 \\ 1 & 0 \end{bmatrix}$. Graph the eigenvalues $\lambda_1(A + kB)$ and $\lambda_2(A + kB)$ for $-8 \leq k \leq 8$. How close does $\lambda_1$ come to $\lambda_2$? $\lambda_1(A + kB)$ are in the first row, first column of the diagonal matrix of eigenvalues, and $\lambda_2(A + kB)$ are in the second row, second column.

**Brief answers**

1. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.25 & 0.25 \\ 0.75 & -0.25 \end{bmatrix}$

2. $A^3 = X\Lambda^3 X^{-1}$, and $A^{-1} = X\Lambda^{-1}X^{-1}$

3. rank $A - 3I$ is 1, not 0. 1 is changed to 0 or any other entry is changed to make two distinct eigenvalues.

4. $\|\lambda\| < 1$; $A_1^k$ does not converge to 0 and $A_2^k$ does converge to 0 as $k \to \infty$.

5. $B^k = X\Lambda X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; multiply to complete the proof.

6. $\lambda = 0, 2, -1$, $N(A) = c \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $N(A - 2I) = c \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $N(A + I) = c \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$,

$$A^k = \frac{1}{3} \begin{bmatrix} 2^{k+1} + (-1)^k & 2^k - (-1)^k & 2^k - (-1)^k \\ 2^k - (-1)^k & 2^{k-1} + (-1)^k & 2^{k-1} + (-1)^k \\ 2^k - (-1)^k & 2^{k-1} + (-1)^k & 2^{k-1} + (-1)^k \end{bmatrix}.$$

7. $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

8. (a)$\lambda_C = 1, -0.8$, $\vec{u}_\infty = c(5, 4)$, $c \neq 0$. (b)$\lambda_D = 1, 0.25, 0.25$, $\vec{u}_\infty = c(1, 1, 1)$, $c \neq 0$.

9. $A = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$, and $A^k \to \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ as $k \to \infty$

10. $\vec{u}_n = \#$ trucks in $\begin{bmatrix} \text{LA} \\ \text{Dallas} \\ \text{Chicago} \end{bmatrix}$ in week n; $\qquad \begin{bmatrix} \text{LA} \\ \text{Dallas} \\ \text{Chicago} \end{bmatrix}_{n+1} = \begin{bmatrix} 0.7 & 0.2 & 0.4 \\ 0.1 & 0.5 & 0.5 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} \text{LA} \\ \text{Dallas} \\ \text{Chicago} \end{bmatrix}_n$;

$\vec{u}_0 = \begin{bmatrix} 1240 \\ 1240 \\ 1240 \end{bmatrix}$; $\qquad \vec{u}_\infty = \begin{bmatrix} 1800 \\ 1140 \\ 780 \end{bmatrix}$

11. $\vec{u}(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

12. $\begin{matrix} r(t) = 20e^{5t} + 10e^{2t} \\ w(t) = 10e^{5t} + 20e^{2t} \end{matrix}$, and $\dfrac{r(t)}{w(t)} \to 2$ as $t \to \infty$.

13. (a) Hint: show $\dfrac{d(v + w)}{dt} = 0$

    (b) $\lambda = 0, -2$, $N(A) = c(1, 1)$, $N(A + 2I) = c(1, -1)$

    (c) $v(1) = 20 + 10e^{-2}$, $w(1) = 20 - 10e^{-2}$, and $v(t) = w(t) = 20$ as $t \to \infty$.

14. $e^{At} = \begin{bmatrix} e^t & 0.5\left(e^{3t} - e^t\right) \\ 0 & e^{3t} \end{bmatrix}$

15. $Q = \dfrac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

16. $A = \dfrac{4}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \dfrac{2}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}$ $\qquad B = \dfrac{25}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix}$

17. False

18. False

19. True

20. False

21. True

22. True