25 points Math 210 Homework 8 Sections 6.5, 7.1-7.4 23 problems

1. For which numbers $b$ and $c$ are these matrices positive definite? $A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$

2. What is the quadratic $f = ax^2 + bxy + cy^2$ for these matrices? Complete the square to write $f$ as a sum of one or two squares $d_1(\,)^2 + d_2(\,)^2$. (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

3. Write $f(x, y) = x^2 + 4xy + 3y^2$ as a difference of squares and find a point $(x, y)$ where $f$ is negative. The minimum is not at $(0, 0)$ even though $f$ has positive coefficients, so the corresponding matrix is not positive definite.

4. Find the 3 x 3 matrix $A$ and its pivots, rank, eigenvalues, and determinant for $f(x_1, x_2, x_3) = 4(x_1 - x_2 + 2x_3)^2$.

5. Which 3 x 3 matrices $A$ and $B$ produce these quadratics?
(a) $\vec{x}^T A\vec{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$. Why is $A$ positive definite?

(b)$\vec{x}^T B\vec{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$. Why is $B$ semidefinite?

6. For what numbers c and d are A and B positive definite?
(a) $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$

7. Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix $A$.

Compute the SVD of the following matrices. Then write the SVD in rank-1 decomposition form.

8. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ 9. $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ 10. $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ 11. $D = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$

12. Find the pseudo-inverses of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ from the last problem; then use them to find the best solution $\vec{x}^+$ for $A\vec{x} = (1, 2)$ and $C\vec{x} = (1, 1, 1)$.

13. Give an example of two different 4 x 4 matrices that have singular values 4, 3, 2, and 1.

14. If $\sigma_1$ and $\sigma_2$ are the singular values for $A = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, what is $\sigma_1^2 + \sigma_2^2$ and how do you know?

Find a 2 x 2 matrix satisfying the given conditions or explain why such a matrix does not exist.

15. $\det(A) = -1$ and $A$ has singular values 1 and 1.

16. $A$ has eigenvalues 1 and 1 with singular values 1 and 2.

Answer "True" or "False" and then defend your answer.

17. Every positive definite matrix is invertible.

18. A diagonal matrix with positive diagonal entries is positive definite.

19. A symmetric matrix with a positive determinant is positive definite.

20. A symmetric matrix cannot be similar to a non-symmetric matrix.

21. An invertible matrix cannot be similar to a singular matrix.

22. If $B$ is invertible then $AB$ is similar to $BA$.

23. CAS Problem (3 points): Use a CAS for the following. Submit a printed copy of your commands and answers. The commands A=rand(20, 40) and B=randn(20, 40) produce 20 x 40 random matrices with entries between 0 and 1 in Matlab. The entries of A are generated using a uniform horizontal line probability distribution and the entries for B are generated using a normal bell-shaped probability distribution. Using a svd command, find and graph the singular values for each of the matrices on the same set of axes. Use different colors for the singular values from A and B. How many singular values will each matrix produce?

**Brief answers**

1. $3 < b < 3$; $c > 8$

2. (a) $x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$ (b) $x^2 + 6xy + 9y^2 = (x + 3y)^2$

3. $f(x, y) = (x + 2y)^2 - y^2$; $f(2, -1) = -1$

4. $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$; pivot is 4, rank = 1, det = 0, $\lambda = 0, 0, 24$.

5. (a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$; pivots $> 0$; (b) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; singular, pivots $\geq 0$.

6. (a) $c > 1$ (b) $B$ is not positive definite for any $d$.

7. Major axis is along $y = -x$ with half length $\sqrt{2}$, and minor axis is along $y = x$ with half length $\sqrt{\dfrac{2}{3}}$.

8. $A = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}} = 5\sqrt{2} \dfrac{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}{\sqrt{10}} \dfrac{\begin{bmatrix} 1 & 2 \end{bmatrix}}{\sqrt{5}}$

9. $B = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix} = \sqrt{3}\frac{\begin{bmatrix}1\\ 1\end{bmatrix}}{\sqrt{2}}\frac{\begin{bmatrix}1 & 2 & 1\end{bmatrix}}{\sqrt{6}} + 1\frac{\begin{bmatrix}1\\ -1\end{bmatrix}}{\sqrt{2}}\frac{\begin{bmatrix}1 & 0 & -1\end{bmatrix}}{\sqrt{2}}$

10. $C = B^T$, so transpose the answer for $B$.

11. $D = \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}5 & 0 & 0\end{bmatrix}\begin{bmatrix}0.6 & 0.8 & 0\\ 0.8 & -0.6 & 0\\ 0 & 0 & 1\end{bmatrix}$

12. $\vec{x}^+ = \begin{bmatrix}0.14\\ 0.28\end{bmatrix}$ and $\vec{x}^+ = \frac{2}{3}\begin{bmatrix}1\\ 1\end{bmatrix}$.

13. For instance, $A = \begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ and $A = \begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$. Many other correct answers exist.

14. $2 + c^2$

15. Exists

16. Does not exist.

17. True

18. True

19. False

20. False

21. True

22. True