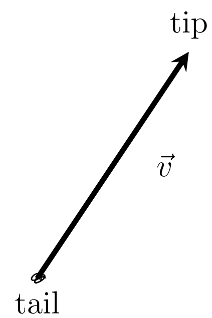
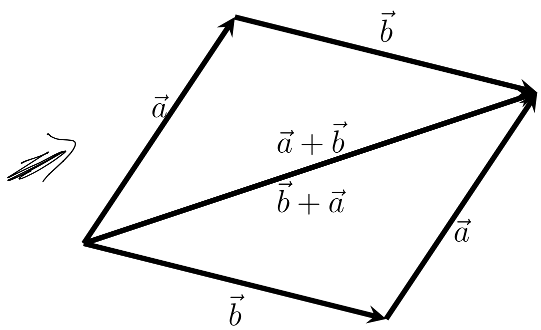
Vectors (HW #1)

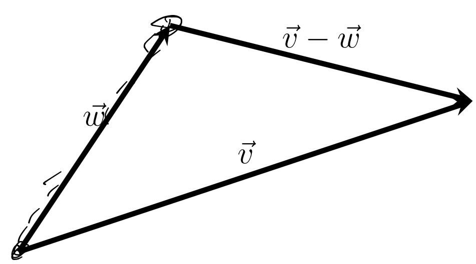
An intuitive definition of a vector is an arrow with direction and length. Notation is $\vec{v}$ or a bold $\mathbf{v}$.



$\|\vec{v}\|$ = the **length** of the vector. **Magnitude** and **norm** are other words for "length."

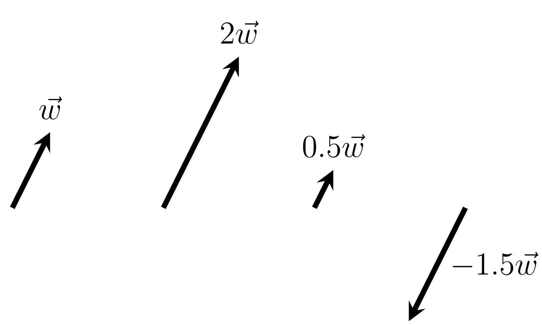
Two vectors are added when the tail of the second is placed at the tip of the first with the sum equal to the vector with tail of the first and the tip of the second. The following pictures show that addition commutes and subtraction translates.





Two vectors are **independent** if they are not parallel. The picture shows two independent vectors determine a parallelogram.

Scalar multiplication is a number multiplied by a vector. It *rescales* the length of the vector, so $\|c\vec{v}\| = |c|\|\vec{v}\|$. Negative numbers reverse direction, but otherwise the direction remains the same. All nonzero scalar multiples are **parallel** vectors.

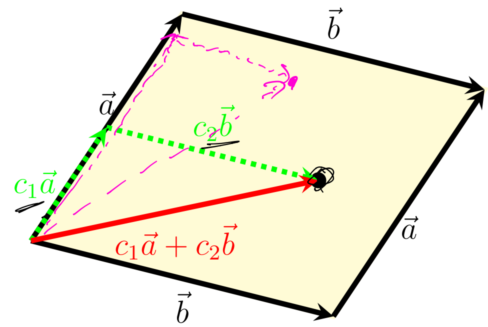


Here is some vocabulary we will use frequently in this class.

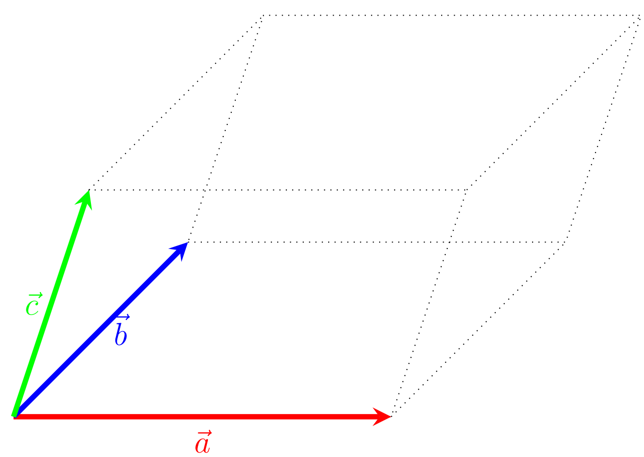
$\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$. Every nonzero vector $\vec{a}$ has a corresponding unit vector in the same direction: $\hat{e}_a = \dfrac{\vec{a}}{\|\vec{a}\|}$. The "hat" notation is frequently used when the vector is a unit vector.

$c_1\vec{a} + c_2\vec{b}$ is a **linear combination** of $\vec{a}$ and $\vec{b}$.

Every point in the parallelogram determined by $\vec{a}$ and $\vec{b}$ is the tip of a linear combination of $\vec{a}$ and $\vec{b}$. We say that the parallelogram is $c_1\vec{a} + c_2\vec{b}$ if both c_1 and c_2 are between 0 and 1.

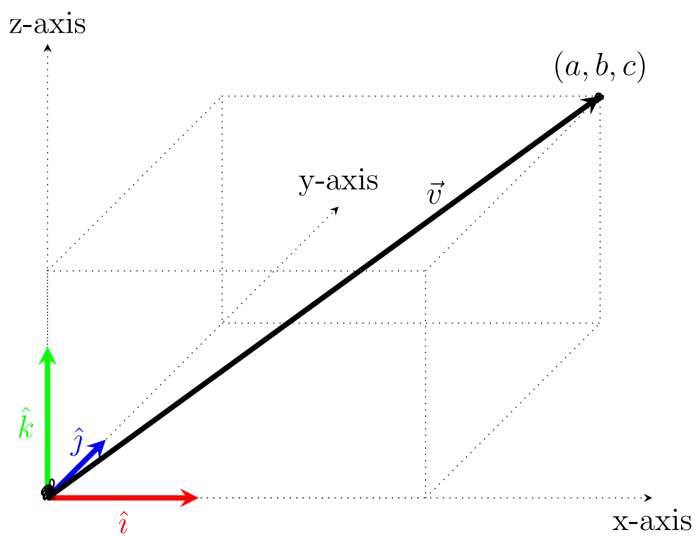


Vectors can be placed into any n-dimensional space, denoted $\mathbb{R}^n$, though we can only draw pictures of them in 1, 2, or 3-space. Here is a picture for $\mathbb{R}^3$:

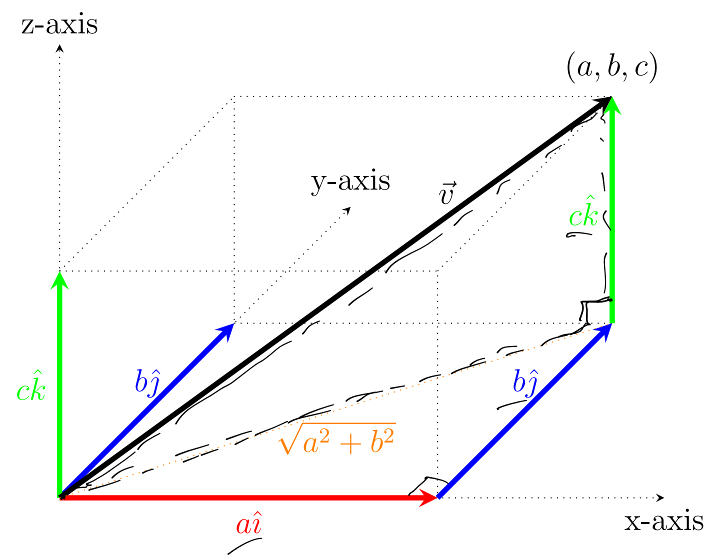


We see that $c_1\vec{a} + c_2\vec{b} + c_3\vec{c}$ is a box (or parallelepiped) in 3-space, again because any point in the box can be determined by a linear combination of those three vectors if $0 \leq c_i \leq 1$.

In order to perform vector arithmetic easily we define vectors in terms of coordinates. To start this process, define unit vectors that are parallel to the coordinate axes. Conventional notation uses $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ to be unit vectors parallel to the x, y, and z-axes respectively. Now position any vector $\vec{v}$ with its tail at (0,0,0). This is called **standard position**. We will assume any vector is in standard position unless specified otherwise.



Now we take a linear combination of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ using coefficients determined by the tip of $\vec{v}$ to decompose $\vec{v}$ into its **coordinate components**.



In math 160 and math 200 we wrote

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \langle a, b, c\rangle$$

where $\langle a, b, c\rangle$ is called the component form for the vector. But in math 210 we write **column vectors** to denote vectors or just use rounded parentheses like a point:

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (a, b, c).$$

Consequently, we have $\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Moreover, using Pythagorean's Theorem twice, we have

$$\|\vec{v}\| = \|(a, b, c)\| = \sqrt{a^2 + b^2 + c^2}.$$

This arithmetic generalizes to $\mathbb{R}^n$ for any positive integer n. If we set $\hat{e}_i$ to be the vector with a 1 in the ith component and zeroes everywhere else, then a vector

$\hat{e}_1 = (1, 0, 0, \ldots, 0)$

$\hat{e}_2 = (0, 1, 0, 0, \ldots, 0)$

$$\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_i \hat{e}_i$$

and

$$\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

For instance, $\vec{x} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 2 \\ 7 \\ 6 \end{bmatrix} \in \mathbb{R}^6$ might represent the voltages in a circuit with six different nodes.

We can prove that scalar multiplication, vector addition, and vector subtraction can be computed component-by-component.

Simplify $\vec{v} = 5(1, -2, 2) - 2(3, 4, -1)$ and then find its length, $\|\vec{v}\|$. Then do the same calculation using Matlab.

$\vec{v} = 5(1, -2, 2) - 2(3, 4, -1)$

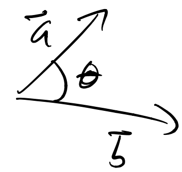
$\vec{v} = (5, -10, 10) - (6, 8, -2)$

$\boxed{\vec{v} = (-1, -18, 12)}$

Dot Product

If θ is the angle between two vectors $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$, then the **dot product** is

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$



and we can prove from this definition that

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i.$$

$(1,2,3) \cdot (-1,3,6) = 1(-1) + 2(3) + 3(6)$
$= -1 + 6 + 18$
$= 23.$

The following properties of the dot product are easy to prove:

1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

2) $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ is **orthogonal** to $\vec{b}$. In $\mathbb{R}^n$ "orthogonal" and "perpendicular" are the same.

3) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

4) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

Compute $(1, 3, 2, -2) \cdot (4, 1, 0, 1)$ by hand and then using Matlab. Matlab uses matrix notation but uses an apostrophe for "transpose" instead of "T". $(1,3,2,-2) \cdot (4,1,0,1) = 1 \cdot 4 + 3 \cdot 1 + 2 \cdot 0 + (-2)(1) = 5$

Matrix notation:

$$(1, 3, 5) \cdot (2, -4, 6) = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}^{T} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = 1 \cdot 2 + 3 \cdot (-4) + 5 \cdot 6$$

"row vector"

$= 2 - 12 + 30 = 20$

The **transpose** of a vector is a **row vector**. So a row vector times a column vector, in that order, is a dot product of the two corresponding vectors.

Equations can be written using dot products and matrix notation:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

is the equation

$$x + 2y + 3z = 0.$$

Repeating three times we can write

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x - 4y - 8z &= 2 \\ 3x + 4y - z &= 5 \end{aligned} \quad \text{as} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & -8 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}.$$