Gaussian Elimination (HW #1)

You have solved systems like $\begin{aligned} 2x + 4y + 6z &= 14 \\ x - 3y + z &= -5 \\ 4x + y - z &= 1 \end{aligned}$ using row operations that allow you to (1) switch the order of the equations (and variables), (2) multiply each row by a constant, and

(3) **replace the ith equation R_i with $R_i - kR_j$ where k is any constant and $i \neq j$.**

Gaussian Elimination uses only the last one. It is called the **elementary row operation**. A computer uses the switching of equations and variables to reduce round-off error, and we will assume that job is all done when we are given a system in math 210.

Gaussian Elimination is the process of producing a lower triangle of zeroes in the system. We will use matrices so that we don't have to write the variables and equal signs over and over again.

$-5 \cdot m = -7 \Rightarrow m = \frac{7}{5}$

Use Gaussian Elimination and matrices to solve $\begin{aligned} 2x + 4y + 6z &= 14 \\ x - 3y + z &= -5 \\ 4x + y - z &= 1 \end{aligned}$. Keep track of the elementary row operations used.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 14 \\ 1 & -3 & 1 & -5 \\ 4 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & -7 & -13 & -27 \end{array}\right] \xrightarrow{R_3 - \frac{7}{5}R_2} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & 0 & -\frac{51}{5} & -\frac{51}{5} \end{array}\right]$$

$-13 - \frac{7}{5}(-2)$

$-13 + \frac{14}{5} = \frac{-65}{5} + \frac{14}{5}$

$-27 - \frac{7}{5}(-12)$

$-\frac{135}{5} + \frac{84}{5} = \frac{-51}{5}$

Now back substitute to solve the system:

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & 0 & -\frac{51}{5} & -\frac{51}{5} \end{array}\right] \end{array} \begin{array}{l} \Rightarrow x = 0 \\ \Rightarrow y = 2 \\ \Rightarrow z = 1 \end{array} \Rightarrow \boxed{(0, 2, 1) \text{ is the only solution}}$$

(3) $\Rightarrow -\frac{51}{5} z = \frac{-51}{5} \Rightarrow z = 1$

(2) $\Rightarrow -5y - 2(1) = -12 \Rightarrow -5y = -10 \Rightarrow y = 2$

(1) $\Rightarrow 2x + 4(2) + 6(1) = 14 \Rightarrow 2x = 0 \Rightarrow x = 0$

The **pivots** for this system are 2, -5, and $-51/5$; pivots are the leading nonzero entries in each row after Gaussian Elimination has been performed and

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & 0 & -51/5 & -51/5 \end{array}\right]$$

is the **row echelon form** (ref) of the system.

Questions this course will answer:

1) If ref $= \begin{bmatrix} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, how do we write the solution set and what does it mean for a circuit?

2) If ref $= \begin{bmatrix} 2 & 4 & 6 & 14 \\ 0 & -5 & -2 & -12 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, what is the "best" solution and how might it influence college admissions?

$0=2 \Rightarrow$ no solution

Two observations will move us towards answering these questions:

1) $\begin{aligned} 2x+4y+6z &= 14 \\ x-3y+z &= -5 \\ 4x+y-z &= 1 \end{aligned}$ can be written using a **linear combination of vectors:**

$$x\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \\ 1 \end{bmatrix}.$$

2) The matrix of coefficients $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & -3 & 1 \\ 4 & 1 & -1 \end{bmatrix}$ can be factored!

$$A = LU = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & 7/5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 4 & 6 \\ 0 & -5 & -2 \\ 0 & 0 & -51/5 \end{bmatrix}}_{U}.$$

L is a **lower triangular matrix** and U is an **upper triangular matrix**. L is formed using the factors from the elementary row operations in Gaussian Elimination. We say the **rank** of A is three, the number of pivots.