# Independent Vectors (HW #1)

A **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a sum of scalar multiples $\sum_{i=1}^{n} c_i \vec{v}_i$ where $c_i$ are scalars.

The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is called the **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

For example, the solution to $\begin{array}{l} 2x + 4y + 6z = 14 \\ x - 3y + z = -5 \\ 4x + y - z = 1 \end{array}$ of $x = 0$, $y = 2$, and $z = 1$ implies that $\begin{bmatrix} 14 \\ -5 \\ 1 \end{bmatrix}$ is in the **span** of

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

and Gaussian elimination showed us

$$0 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

is the only **linear combination** of those vectors that equals

$$\begin{bmatrix} 14 \\ -5 \\ 1 \end{bmatrix}.$$

More examples:

span $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, c \in \mathbb{R}$, is a line through $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

span $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}, c_1, c_2 \in \mathbb{R}$, is the same line.

Proof: $c_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + 2c_2 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = (c_1 + 2c_2) \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ since both $c$ and $c_1 + 2c_2$ can be any real number.

$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$ are **dependent** vectors.

$$\text{span}\left\{\begin{bmatrix}2\\1\\4\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\} = c_1\begin{bmatrix}2\\1\\4\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\end{bmatrix}, c_1, c_2 \in \mathbb{R}, \text{ is a plane through } \vec{0}.$$

$\begin{bmatrix}2\\1\\4\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$ are **independent** vectors because neither vector is in the span of the other.

span $\begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}8\\4\\4\end{bmatrix}$ is the same plane through $\vec{0}$ because

$$1\begin{bmatrix}2\\1\\4\end{bmatrix} + 3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}8\\4\\4\end{bmatrix}.$$

So $\begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}8\\4\\4\end{bmatrix}$ are dependent vectors because one is in the span of the others.

A set of vectors is **independent** if none of the vectors is in the span of the others.

This is a very important concept, so we will give two more equivalent definitions.

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is **linearly independent** if and only if the system of equations $\left[\vec{v}_1 \quad \vec{v}_2 \quad \ldots \quad \vec{v}_n \quad | \quad \vec{0}\right]$ has only the **trivial** solution $\vec{x} = \vec{0}$.

Are $\begin{bmatrix}-2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}$ independent or dependent?

Are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ independent or dependent?

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is **linearly independent** if and only if the matrix made by putting the vectors in columns $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$ has n pivots.

It is especially important in this class to memorize all definitions; please be especially vigilant about memorizing these three equivalent definitions of linear independence.