Invertible Matrices (HW #2)

Let A be an n x n matrix, and suppose there is an n x n matrix A^{-1} so that $A^{-1}A = I$. Then $A\vec{x}_i =$(col $i\, I$) implies $\vec{x}_i = A^{-1}$ (col $i\, I$) = (col $i\, A^{-1}$), so $AA^{-1} = I$. In a similar way, using rows instead of columns, we can prove that $AA^{-1} = I$ implies $A^{-1}A = I$. That means $AA^{-1} = I \iff A^{-1}A = I$ so that the following definition valid.

An n x n matrix A is **invertible** with inverse A^{-1} if and only if $AA^{-1} = A^{-1}A = I$.

The inverse of A is unique: if $AB = I$, then

$$B = IB = A^{-1}AB = A^{-1}I = A^{-1}.$$

Consequently, to determine if A is invertible, we need only find an inverse for it. Often you can find the inverse by inspection using properties of matrix multiplication.

True or False? If A and B are invertible, then so is AB.

True or False? If A is n x n and $A^2 = [0]$, then $I - A$ is invertible.

An n x n matrix A is **nonsingular** if and only if the system of equations $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.

If A is invertible, then $A\vec{x} = \vec{b}$ has unique solution $\vec{x} = A^{-1}\vec{b}$, so A is nonsingular.

On the other hand, if A is nonsingular, then $A\vec{x}_i = \hat{e}_i$, where $\hat{e}_i$ is col i of I, has a solution for each i which implies the matrix with i^{th} column equal to $\vec{x}_i$ is the inverse of A and so A is invertible.

So A is invertible if and only if it is nonsingular. In particular, if A is invertible, then $A\vec{x} = \vec{0}$ has a unique solution, namely $\vec{0}$, which means the only linear combination of the columns of A equal to to $\vec{0}$ is the trivial one, and so

the columns of an invertible matrix are linearly independent!

Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ invertible or singular?

How do we find the inverse when a matrix is invertible?

As said earlier, some matrices have inverses that are easy to find using properties of multiplication.

Elementary Matrices. Find the matrix that undoes the elementary row operation. Find the inverse for $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Matrices with orthogonal columns or rows. Find the inverse for $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

2 x 2 matrices with linearly independent columns. The inverse for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$. Verify this. What is the inverse of $K = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$?

It is most efficient to use the LU-factorization to find the inverse of a large matrix, but many textbooks emphasize using "Gauss-Jordan elimination" for small matrices. Let's use that technique to find the inverse of $M = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 1 \\ 1 & 6 & 7 \end{bmatrix}$. This technique solves the three systems $M\vec{x} = \vec{e}_i$ all at once.