Basis and Fundamental Spaces (HW #3)

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a linearly independent (or "independent" for short) set of vectors iff
(Give three definitions.)

If "subvectors" are independent, then so are the vectors: for example, we can tell that $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 7 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ is an independent set of vectors at a glance. How?

But then the method we use when finding the null space of a matrix guarantees the "special solutions" are independent. Let $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What is $N(A)$?

The set of "special solutions" are thus (1) linearly independent and (2) span the vector space $N(A)$. When those two conditions are true, we call the set of vectors a **basis** for $N(A)$. We can now replace the unconventional name "special solutions" with the conventional name: basis. The number of vectors in a basis for a vector space is called the **dimension** of the vector space.

The **standard basis** for $\mathbb{R}^4$ is

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \hat{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

What is the dimension of $\mathbb{R}^4$? Generalize to $\mathbb{R}^n$

Every vector space has an infinite number of bases. Find ten different bases for $\mathbb{R}^2$. "bases" is plural for "basis."

To reiterate:

A **basis** of a vector space is a set of vectors in that space that
1) is linearly independent, and
2) spans the vector space; that is, the set of all linear combinations equals the vector space.

The **dimension** of a vector space is the number of vectors in any basis of that vector space.

$V = \{(x, y, y, y)|x, y \in \mathbb{R}\}$ is a vector space in $\mathbb{R}^4$. Find a basis and the dimension for V.

Is $P =$ all polynomials with degree less than or equal to two a vector space? If so, find a basis and its dimension.

Our definition of dimension assumes that every basis of the same vector space has the same number of vectors. That needs to be proved.

Theorem: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a basis for a vector space V, then every basis of V has n vectors.

Proof: Suppose two sets of independent vectors have the same span. If we prove that the set with the larger number of vectors is dependent, we have a contradiction. Thus two sets of independent vectors with the same span would have to have the same number of vectors and our proof would be finished. Instead of proving this in general, I will prove it for a small case.

Suppose span $\vec{v}_1, \vec{v}_2 =$ span $\vec{w}_1, \vec{w}_2, \vec{w}_3$. I want to show that $\vec{w}_1, \vec{w}_2, \vec{w}_3$ is a dependent set if $\vec{v}_1, \vec{v}_2$ is independent. To do this, I will prove there is a nontrivial linear combination of $\vec{w}_1, \vec{w}_2, \vec{w}_3$ that equals zero. Because the spans are equal, we know there are scalars a_{ij} for which

$$\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2$$

$$\vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2$$

$$\vec{w}_3 = a_{13}\vec{v}_1 + a_{23}\vec{v}_2.$$

But then

$$c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3 = \vec{0}$$

$$\implies (c_1a_{11} + c_2a_{12} + c_3a_{13})\vec{v}_1 + (c_1a_{21} + c_2a_{22} + c_3a_{23})\vec{v}_2 = \vec{0}$$

after substituting and collecting like terms. Since the set $\vec{v}_1, \vec{v}_2$ is independent, the coefficients in the above equation must be zero and so we can write the system of equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But the 2 x 3 matrix has at most two pivots and so at least one free variable meaning there is a nontrivial solution for the c_i. Then $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are dependent as we were to show.

Fundamental Spaces of a Matrix A

The **fundamental spaces** of a matrix A are $N(A)$, the null space; $N(A^T)$, the left null space; $C(A)$, the column space; and $C(A^T)$, the row space. We can find the LU factorization of A (or PA where P is a permutation - which will give an equivalent result) and from that we can find a basis for each fundamental space.

Find a basis and the dimension for each fundamental space of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Notice that the number of pivot columns in U equals the number of column vectors in L that form a basis for $C(A)$ **and** equal the number of nonzero rows in U that make a basis for $C(A^T)$. Therefore $\dim C(A) = \dim C(A^T) = r$, where r is the rank of A. Furthermore, The dimension of $N(A)$ equals the number of free columns, so if A is m x n,

$$\dim C(A^T) + \dim N(A) = n.$$

Similarly,

$$\dim C(A) + \dim N(A^T) = m.$$

This is the first part of what Gilbert Strang (the author of your textbook) calls the **Fundamental Theorem of Linear Algebra**, or FTLA for short. The picture on the front cover of your book depicts the full statement of that theorem[1].

Is there a matrix A for which $(1,2,1)$ and $(1,3,1)$ are in $N(A)$ and $(0,2,1)$ and $(1,1,1)$ are simultaneously in $C(A)$? Defend your answer by constructing such a matrix or proving that it can't be done.

[1]Most other authors call this the dimension theorem and do not attach it with the orthogonality statement of the second part of the theorem.

As time allows.

Write down a basis and the dimension for each fundamental space of $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$ without doing any work.

Find a basis and the dimension for each fundamental space of $K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 4 \\ 0 & 2 & 1 & 2 \end{bmatrix}$.