Orthogonality and the FTLA (HW #4)

A set of vectors S is **orthogonal** to another set of vectors W if every vector in S is orthogonal to every vector in W. That is, if $\vec{x} \in S$ and $\vec{y} \in W$, then $\vec{x} \cdot \vec{y} = 0$.

If S is the set of all vectors parallel to the xy-plane and W is the set of all vectors parallel to the yz-plane, are S and W orthogonal sets of vectors? Defend your answer.

If S is a set of vectors, then its **orthogonal complement** $S^{\perp}$ is the set of all vectors orthogonal to all vectors in S.

True or False: $S^{\perp}$ is a vector space. Defend your answer.

If A is any matrix, then $\vec{x} \in C(A^T)^\perp \implies \vec{x}$ is orthogonal to the rows of A so that $\vec{x} \in N(A)$. But is every vector in $N(A)$ also in $C(A^T)^\perp$? The answer is "yes" and here is an illustration of why (it is not a proof!)

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Verify that $\vec{x} = (1, -2, 1) \in N(A)$. Notice how it is orthogonal to each row vector of A. Now create any linear combination of the rows of A and find the dot product with $\vec{x}$.

We see that any vector in $N(A)$ must be orthogonal to any linear combination of the rows of A. But then $N(A) = C(A^T)^\perp$. Replacing A with A^T also gives us $N(A^T) = C(A)^\perp$.

FTLA can now be stated in full.

If A is an m x n matrix and r is the rank of A, then

$$\text{dim}C(A) = \text{dim}C(A^T) = r$$

$$\text{dim}N(A) + \text{dim}C(A^T) = \text{n}$$

$$\text{dim}N(A^T) + \text{dim}C(A) = \text{m}$$

$$N(A) = C(A^T)^\perp$$

$$N(A^T) = C(A)^\perp$$

True or False? Defend your answer.

1) There is a matrix A with $(1,2,3)$ in $C(A^T)$ and $(1,1,1)$ in $N(A^T)$.

2) There is a matrix B with $(1,2,3)$ in $C(B^T)$ and $(1,1,1)$ in $N(B)$.

3) There is a nonzero matrix C with every row orthogonal to every column.

4) There is a matrix M for which the columns add to a column of zeroes and the rows add to a row of ones. Hint: think two ways - multiplication is a combo of columns and FTLA.

As time allows.
It is true that if V is a vector subspace of a finite dimensional vector space, then $(V^{\perp})^{\perp} = V$. The proof is easy after we learn about projection onto vector spaces, our next topic. Show this is not true if V is not a vector subspace.