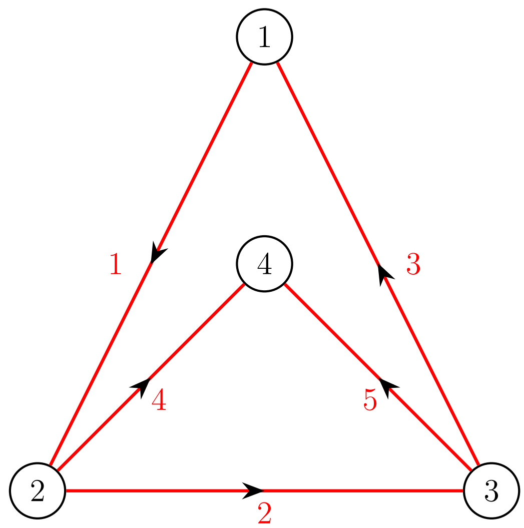
Application of FTLA: Circuits (HW #4)

This material is in Section 10.1 of the 5th edition of your textbook.



A circuit is represented above by a directed graph with four nodes in black and five edges in red. Electrons flow from one node to the other if the voltages (or electric potentials) are different. The unknowns are:

1) the node voltage vector $\vec{x} = (x_1, x_2, x_3, x_4)$ consisting of the voltages x_i at node i, and

2) the edge current vector $\vec{y} = (y_1, y_2, y_3, y_4, y_5)$ consisting of the currents y_i amps at edge i.

The problem is to find these unknown vectors if we are given the directed graph and:

1) the edge conductances c_i on edge i. They will become the diagonal elements of a diagonal square conductance matrix C. Conductance $= 1/$Resistance, so since Resistance is given in "ohms" with symbol Ω the units of Conductance are "mhos" with symbol ℧.

2) the external node current vector $\vec{f} = (f_1, f_2, f_3, f_4)$ of the external currents f_i **flowing into** node i

To solve the problem, we first must find an expression for the change in voltage along an edge - this causes the flow of electricity. The voltage **gain** for directed edge 1, for instance, is $x_2 - x_1$. We can write that as a matrix product

$$\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} \vec{x}.$$

For this reason, we decide $\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$ will be the first row in the **incidence matrix** A representing the first edge of the circuit. The columns of this matrix will represent the nodes, and the rows will represent the edges.

Write down the rest of A and verify that $A\vec{x}$ is the voltage gain vector on each edge. Since $A\vec{x}$ is a linear combination of columns, $C(A)$ must be the space of all voltage gain vectors.

Kirchoff's Voltage Law (KVL) says the voltage gain around a loop equals zero. A loop is a path along edges that begins and ends at the same node. Each loop has its own orientation separate from the orientation on the edges. Linear combinations of the edges produce paths. Find a linear combination of edges that form a loop.

Now write that path as matrix multiplication involving a "loop" vector and A. What fundamental space contains the "loop" vector?

We see then that KVL is satisfied by the way we constructed A. We have also discovered $N(A^T)$ is the space of loop vectors. Find a basis for $N(A^T)$. This could consist of the "inner loops."

$C(A)$ is orthogonal to $N(A^T)$, so $A\vec{x}$ is orthogonal to every loop vector. Can $A\vec{x} = (1, 2, 3, 4, 5)$ in our example? Defend your answer.

What do the other fundamental spaces of A mean in the context of the circuit?

$A\vec{x} = \vec{0}$ implies the voltage gains are all 0, so the node voltages must be the same on every edge. We assume the circuit is "path connected" - each node has a path to any other, so that means all the nodes must have the same voltage. Consequently,

$$N(A) = c\vec{1}.$$

$C(A^T) = \{A^T\vec{y} \,|\, \vec{y} \in \mathbb{R}^5\}$ is the net flow into nodes from the internal edges. This is related to Kirchoff's Current Law (KCL) which says that a circuit is in equilibrium when the net current flow into any node is 0. Multiply the first row of A^T by $\vec{y}$ and relate the result to the graph. If time allows and there is a desire, multiply all of A^T by $\vec{y}$ and relate the result to the graph.

Notice that $A^T\vec{y} = -\vec{f}$ implies $\vec{f}$ is contained in the row space of A. Using our 5 x 4 incidence matrix A, can $\vec{f} = (1, 2, 3, 4)$? Defend your answer.

Ohm's Law says (voltage **drop**) = (Current)(Resistance), but we will write it intstead as (Conductance)(voltage **drop**) = Current.

Remembering that $\begin{bmatrix}-1 & 1 & 0 & 0\end{bmatrix}\vec{x}$ is a voltage **gain** for edge 1, Ohm's Law for the first edge is

$$c_1\left(-\begin{bmatrix}-1 & 1 & 0 & 0\end{bmatrix}\vec{x}\right) = y_1$$

.

For all edges, Ohm's law becomes

$$-CA\vec{x} = \vec{y} \qquad (*)$$

.

Now apply the Kirchoff Current Law (KCL) that says the net current flow into each node is zero. KCL is expressed as (currents into a node from inner edges) + (external current into a node) = 0, or, using matrices to express this for all nodes, $A^T\vec{y} + \vec{f} = 0$ which can be written as

$$A^T\vec{y} = -\vec{f} \qquad (**)$$

$(*)$ and $(**)$ together make a system that can be solved by substitution giving us

$$A^TCA\vec{x} = \vec{f}$$

.

Let's solve this problem using Matlab with $c_i = 2i$ and $\vec{f} = (3, 1, -2, -2)$. Look at the matlab notes for homework 4.

Notice how A^TCA is symmetric (can you prove it?), and that

$$(A^TCA)_{ij} = \begin{cases} 0 & \text{no edge between node } i \text{ and } j \text{ for } i \neq j \\ -c & c \text{ is conductance of edge between node i and j} \\ \sum c\text{ 's into node } i & i = j \end{cases}$$

It is also true that $N(A^TCA) = N(A)$.[1] So $N(A^TCA) = c\vec{1}$ implies it has one free variable and so we can set the voltage of any node to 0. This is called "grounding a node."

[1] $\vec{x} \in N(A) \implies A^TCA\vec{x} = A^TC\vec{0} = \vec{0} \implies \vec{x} \in N(A^TCA)$. Also, $\vec{y} \in N(A^TCA) \implies$ the dot product $(\sqrt{C}A\vec{y})^T(\sqrt{C}A\vec{y}) = 0 \implies \sqrt{C}A\vec{y} = \vec{0} \implies A\vec{y} = \vec{0} \implies \vec{y} \in N(A)$.

Solve this problem by hand. $c_1 = 2$, $c_2 = 1$, $c_3 = 5$; $f_1 = 5$, $f_2 = 1$, $f_3 = -6$, and the directed graph is below. Ground the third node. Once you have the solution, verify that KCL and KVL are satisfied, and then find a basis for each space and relate it to the graph.