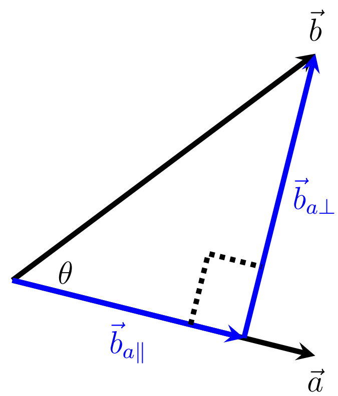
Projection (HW #4)

This material is in Section 4.2 of the 5th edition of your textbook.

Every vector $\vec{b}$ can be decomposed with respect to any other vector $\vec{a}$ into a parallel component and a perpendicular component.

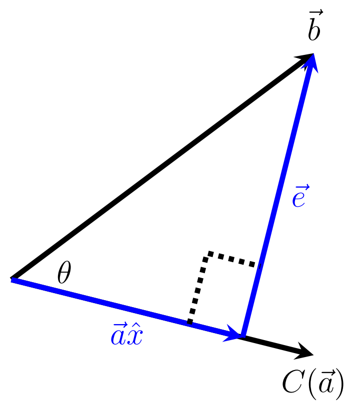


In math 200 and math 160, $\vec{b}_{a\|}$ is the **projection of $\vec{b}$ onto $\vec{a}$** and $\vec{b}_{a\perp}$ is the **perpendicular component of $\vec{a}$ to $\vec{b}$**. Notice $\vec{b} = \vec{b}_{a\|} + \vec{b}_{a\perp}$ and so if we know two of the three vectors, we can find the third. We used trigonometry to prove that

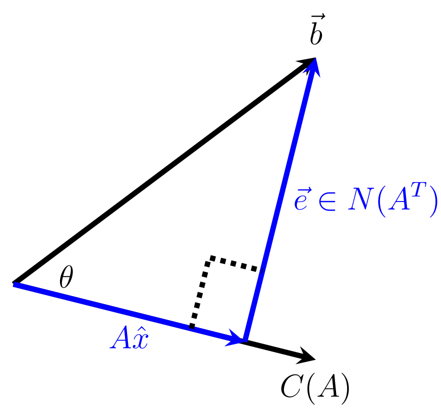
$$\vec{b}_{a\|} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\,\vec{a}$$

Find $\vec{b}_{a||}$ if $\vec{b} = (1, 1, 1)$ and $\vec{a} = (1, 2, 2)$.

In math 210, I call $\vec{b}_{a\perp}$ the **error vector** and denote it by $\vec{e}$. The projection is the vector in $C(\vec{a})$ that best estimates $\vec{b}$: the Pythagorean theorem tells us any other vector has a longer error vector. $\hat{x} = \dfrac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}$ is the scalar multiple so that the projection is $\vec{a}\hat{x}$.



We wish to generalize this concept so that we can project $\vec{b}$ onto any $C(A)$. To do this, interpret the last picture in terms of the fundamental spaces of the matrix A.



This picture says the system $A\vec{x} = \vec{b}$ does not have a solution. Instead, we find the "best" solution by solving $A^T(\vec{e}) = \vec{0} \implies A^T(\vec{b} - A\hat{x}) = \vec{0}$ or solving

$$A^T A\hat{x} = A^T\vec{b}$$

for $\hat{x}$. As a shortcut, I think of trying to solve $A\vec{x} = \vec{b}$ by multiplying both sides by A^T; but in this case we don't have the same unknown, $\hat{x}$ instead of $\vec{x}$, and so get the best approximation to the answer instead of an exact answer.

Find the projection of $\vec{b} = (1, 1, 1)$ onto the span of $(1, 2, 2), (0, 1, 0)$.

If the columns of A are independent, then $\vec{0} = N(A) = N(A^T A)$[1] and so $A^T A$ is invertible. Then $A^T A\hat{x} = A^T\vec{b} \implies \hat{x} = (A^T A)^{-1}A^T\vec{b}$ so that

$$A\hat{x} = A(A^T A)^{-1}A^T\,\vec{b}.$$

We call the matrix $P = A(A^T A)^{-1}A^T$ the projection matrix onto $C(A)$. Show that P is symmetric and that $P^2 = P$. This last equation means once projected, the vector won't move if projected again.[2]

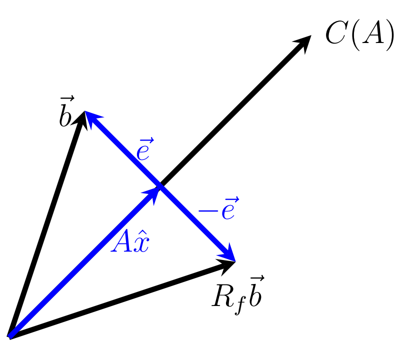
[1] $\vec{x} \in N(A) \implies A^T A\vec{x} = A^T\vec{0} = \vec{0} \implies \vec{x} \in N(A^T A)$. $\vec{y} \in N(A^T A) \implies A^T A\vec{y} = \vec{0} \implies \vec{y}^T A^T A\vec{y} = 0 \implies \|A\vec{y}\|^2 = 0 \implies A\vec{y} = \vec{0} \implies \vec{y} \in N(A)$.

[2] Projection matrices are often defined to be any matrix satisfying $P^T = P$ and $P^2 = P$. Given such a matrix, how do we know what column space is being projected onto?

Find the projection matrix onto the span of $(1, 2, 2), (0, 1, 0)$. Verify that $P\vec{b}$ is the projection we calculated in the last example.

AS TIME ALLOWS

Assuming P is the projection matrix onto $C(A)$, what matrix R_f **reflects** across $C(A)$?



Since $P\vec{b} = A\hat{x}$, $R_f\vec{b} = P\vec{b} - \vec{e}$. Now find R_f in terms of P.