Orthonormal Basis, QR-Factorization (HW #5)

This material is in Section 4.4 of the 5th edition of your textbook.

If $\hat{u}$ is a unit vector ($||\hat{u}|| = 1$) then the Fourier coefficient of the projection is just a dot product:

$$\vec{b}_{||\hat{u}} = \frac{\vec{b} \cdot \hat{u}}{\hat{u} \cdot \hat{u}} \hat{u} = (\vec{b} \cdot \hat{u})\hat{u}.$$

If we have two unit vectors that are **orthogonal** then projection onto their span is also easier than before. For example, suppose $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \\ 0 & 0 \end{bmatrix}$ where θ is constant. Notice the two column vectors are unit vectors that are orthogonal to each other. Project $(1, 2, 3)$ onto the column space of A.

Then $\hat{x} = \begin{bmatrix} (\cos(\theta), \sin(\theta), 0) \cdot (1, 2, 3) \\ (-\sin(\theta), \cos(\theta), 0) \cdot (1, 2, 3) \end{bmatrix}$,

so each component is a dot product of $(1, 2, 3)$ with the corresponding column vector.

Then $A\hat{x} = \hat{x}_1(\cos(\theta), \sin(\theta), 0) + \hat{x}_2(-\sin(\theta), \cos(\theta), 0)$ is **just the sum of projections onto each column vector**.

Columns made of unit vectors that are mutually orthogonal make projection easy!!!

Here are some definitions that will make the preceding information more concise.

A set of vectors $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ are **orthonormal** if and only if $\vec{q}_i \cdot \vec{q}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Theorem: An orthonormal set of vectors is independent. We have proved something similar to this before. Can you prove it now?

A **square** matrix with orthonormal columns is an **orthogonal** matrix.

Is $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \\ 0 & 0 \end{bmatrix}$ an orthogonal matrix?

Theorem: If Q is an orthogonal matrix, then its inverse is Q^T. We have proved something like this before. Can you prove it now?

In a similar fashion we can show that if the columns of an m x n matrix A are orthonormal, then m $\geq$ n and $A^T A = I$.

How can we find an orthonormal basis for any column space? Suppose we have independent columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$.

The **Gram-Schmidt Method** makes an orthonormal basis $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ from the columns of A in this way:

$\vec{q}_1 = \dfrac{\vec{a}_1}{||\vec{a}_1||}$. Then $\vec{q}_2 = \dfrac{\vec{e}_2}{||\vec{e}_2||}$ where $\vec{e}_2$ is the error vector found when projecting $\vec{a}_2$ onto $\vec{q}_1$.

Then we repeat the process for the projection of $\vec{a}_3$ onto the span of $\vec{q}_1$ and $\vec{q}_2$. Recall how easy it is to project onto that span.

Recursively,

$$\vec{e}_n = \vec{a}_n - (\vec{a}_n \cdot \vec{q}_1)\vec{q}_1 - (\vec{a}_n \cdot \vec{q}_2)\vec{q}_2 - \cdots - (\vec{a}_n \cdot \vec{q}_{n-1})\vec{q}_{n-1}$$

and

$$\vec{q}_n = \frac{\vec{e}_n}{||\vec{e}_n||}.$$

Find an orthonormal basis for $C(A)$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. In this example, $\vec{a}_1 = (1, 1, 1, 1)$, $\vec{a}_2 = (1, -1, 1, 1)$, and $\vec{a}_3 = (1, 1, -1, 1)$.

Notice that $\vec{a}_k$ is in the span of $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k$, so if we project $\vec{a}_k$ onto this span the $\vec{q}_n$ for $n > k$ are not involved and the projection is exactly equal to $\vec{a}_k$. Thus

$$\vec{a}_k = (\vec{a}_k \cdot \vec{q}_1)\vec{q}_1 + (\vec{a}_k \cdot \vec{q}_2)\vec{q}_2 + \cdots + (\vec{a}_k \cdot \vec{q}_k)\vec{q}_k$$

which, when written in matrix form gives the QR factorization of A. This factoriztion is much more useful than the LU factorization, but that topic is beyond the scope of our course. Instead, I'll merely ask you to know how to find it.

$$QR = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \ldots & \vec{q}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 \cdot \vec{q}_1 & \vec{a}_2 \cdot \vec{q}_1 & \ldots & \vec{a}_n \cdot \vec{q}_1 \\ 0 & \vec{a}_2 \cdot \vec{q}_2 & \ldots & \vec{a}_n \cdot \vec{q}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \vec{a}_n \cdot \vec{q}_n \end{bmatrix}$$

Find the QR factorization for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

As Time Allows:

Find the QR factorization for Find the QR factorization for $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

What will happen if one of the columns is dependent on the others?