Fourier Series (HW #5)

This material is in Section 10.5 of the 5th edition of your textbook.

The Fourier series of a **periodic** function is a **projection** onto a particular vector space of continuous periodic functions that is the **span of an orthogonal basis**.

It allows us to, among many other things, mechanically transfer sound into waves and then back to sound, so it makes your cell phones work. Matlab does this process with the Fast Fourier Transform (FFT) and the inverse FFT.

The vector $(2, 5)$ has a finite number of components, a first and second one. The components are 2 and 5 respectively.

A function $f(x)$ with real domain has an infinite number of components, one for each number in the domain. The components themselves are the outputs - for instance, $f(3.2)$ is the 3.2^{th} component of f.

But we know a projection needs a dot product - what is the dot product of a vector space of functions? It is a generalization of the dot product called a real **inner product**. The notation is $\langle f, g \rangle$ and could be defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

There are other possible inner products. They must have the same properties as a dot product[1].

Our vector space for this class will be a space of 2π-periodic functions. I will denote it as $\mathscr{F}_{2\pi}$ and define it to equal the span of $B = \{1, \cos(nx), \sin(nx) \,|\, n = 1, 2, 3, \ldots\}$.

Restating another way, $\mathscr{F}_{2\pi}$ is the vector space equal to all linear combinations of the **function** 1, and $\cos(x)$, $\cos(2x)$, $\cos(3x)$, $\cos(4x)$, $\ldots$, and $\sin(x)$, $\sin(2x)$, $\sin(3x)$, $\sin(4x)$, $\ldots$.

To make calculations easy, I will use the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

if f and g are both in $\mathscr{F}_{2\pi}$.

[1]the properties are the following: 1) $< f, g > = < g, f >$; 2) $< kf, g > = k < f, g >$ if k is a constant; 3) $< f + g, h > = < f, h > + < g, h >$; 4) $\| < f, f > \| \geq 0$ and equals 0 if and only if $f(x) = 0$.

B is an orthogonal and **almost** orthonormal basis. If you remember your trig identities the proof is easy; explaining how to derive all the trig identities takes too long. I expect you to know B is orthogonal and how close it is to being orthonormal, but not prove it on a test or quiz. Here is the proof below which you may read at your own leisure.

$< \cos(nx), \cos(nx) > = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(nx)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}\, dx = 1.$ $n = 1, 2, 3, \ldots.$

$< \sin(nx), \sin(nx) > = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(nx)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\, dx = 1.$ $n = 1, 2, 3, \ldots.$

$< 1, 1 > = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx = 2.$ So the basis B has all unit vectors except for (ironically) 1. The magnitude of the **function** 1 with this inner product is $\sqrt{2}$.

$< \cos(mx), \cos(nx) > = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos((m+n)x) + \cos((m-n)x)}{2}\, dx = 0$ if $m \neq n$ and both m and n can be $0, 1, 2, \ldots$ so that the function 1 is covered as well.

$< \sin(mx), \sin(nx) > = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2}\, dx = 0$ if $m \neq n$.

$< \sin(mx), \cos(nx) > = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\, dx = 0$ if m and n can be $0, 1, 2, \ldots$ so that the function 1 is covered as well.

The last three integrals show the basis B is orthogonal and hence a basis - we have proved any set of orthogonal vectors are linearly independent and the span of B is by definition $\mathscr{F}_{2\pi}$.

What is $\langle 1 + 2\sin(x) - 4\cos(3x), \sin(2x) + 8\cos(3x) - \cos(10x)\rangle$?

The **Fourier series**, $\hat{f}$, of a 2π-periodic function $f(x)$ (it may not be continuous!) is the projection of $f(x)$ onto $\mathscr{F}_{2\pi}$. Since the basis is orthogonal, the projection is just the sum of projections onto each basis element:

$$\hat{f}(x) = \frac{< f, 1 >}{2} + \sum_{n=1}^{\infty} \left[\langle f, \cos(nx) \rangle \cos(nx) + < f, \sin(nx) > \sin(nx) \right].$$

Conventionally a_n is used for the even components and b_n is used for the odd components, so that

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right].$$

Find the Fourier series of a 2π-periodic function $f(x) = x^3$ for $-\pi \leq x < \pi$.

If we define the angle between two functions using the inner product as we would the dot product, what is the angle between $f(x)$ and $\sin(x)$?

Here is a picture of some discrete Fourier series approximations of f.