Properties of Determinants (HW #6)

This material is in Section 5.1 of the 5th edition of your textbook.

The determinant of a 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. We wish to extend this definition to any n x n matrix. We observe that the 2 x 2 determinant definition satisfies these three properties:

1) **Identity**. det(I)=1.

2) **Orientation**. Interchanging two rows changes the sign of the determinant.

For instance, $50 = \begin{vmatrix} 10 & 5 \\ 20 & 15 \end{vmatrix} = -\begin{vmatrix} 20 & 15 \\ 10 & 5 \end{vmatrix}$.

Notice further if two rows are equal, then this property tells us that $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ a & b \end{vmatrix} \implies \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$.

3) **Multilinearity**.

(3a) det distributes over addition of any single row.

For instance, $50 = \begin{vmatrix} 10 & 5 \\ 20 & 15 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 20 & 15 \end{vmatrix} + \begin{vmatrix} 8 & 4 \\ 20 & 15 \end{vmatrix}$

(3b) If B=A except that (row i of B)=k(row i of A) where k is a constant, then $\det(B) = k\det(A)$.

For instance, $50 = \begin{vmatrix} 10 & 5 \\ 20 & 15 \end{vmatrix} = 5\begin{vmatrix} 2 & 1 \\ 20 & 15 \end{vmatrix} = 25\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

Furthermore, if we *define* $\det(A)$ to be the number that satisfies these three properties, then we can prove $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Do this.

Consequently for a n x n matrix A, we define $\det(A)$ to be the number that satisfies the Identity, Orientation, and Multilinearity properties.

From that definition we can prove the following properties. Give examples for each.

4) **If two rows of A are dependent, then $\det(A) = 0$.**

5) **Changing A using an elementary row operation will not change the determinant.** Use this property to find $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$.

6) **Triangular matrices (and hence diagonal ones) have determinant equal to the product of their diagonal entries.** Notation: If T is triangular, then $\det(T) = \prod T_{ii}$. Using this property, if $A = LU$, what is $\det(U)$?

7) **The product rule:** $\det(AB) = \det(A)\det(B)$**.** Consequently, if $A = LU$ is the LU-factorization of A, then $\det(A) = \det(U)$. If A is singular, then U has a row of zeroes, and so by the multilinearity property the determinant is 0. The determinant is the product of the pivots!

Use this property to find $\det(A^{-1})$ in terms of invertible A.

I will somewhat repeat the book's proof of the product rule at the end of the notes for those interested.

8) **The determinant of A^T equals that of** A. $\det(A^T) = \det(A)$. So all that is true about rows of determinants is also true about the columns! Prove this for the special case $A = LU$.

Use properties 8) and 9) to show that if Q is orthogonal, then $\det(Q) = \det(Q^T) = \pm 1$.

Find $|A| = \begin{vmatrix} 0 & 2 & 3 \\ 4 & 5 & 5 \\ 2 & 1 & 3 \end{vmatrix}$ using the properties we have developed. Do not use techniques that you have learned from other classes in the past.

As time Allows.

The proof of property 7) given in the textbook is quite nice.

Let $D(A) = \dfrac{\det(AB)}{\det(B)}$. The proof will be finished if we show $D(A) = \det(A)$. We do this by showing $D(A)$ satisfies properties 1, 2, 3a, and 3b.

$D(I) = \dfrac{\det(B)}{\det(B)} = 1$, so property 1 is satisfied.

Switching two rows of A means we switch the same two rows of AB since (row i AB) = (row i A)B. Then $D(A)$ switches sign because $\det(AB)$ does. So $D(A)$ satisfies property 2.

Using (row i AB) = (row i A)B again, multiplying a row of A by k does the same for AB, so $D(A)$ changes by a factor of k and so satisfies property 3b.

Finally, (row i A_1B+row i A_2B) = $\Big[$(row i A_1)+(row i A_2)$\Big]B$, so $D(A) = D(A_1) + D(A_2)$ if A_1 and A_2 equal A in all rows except in row i and the sum of row i for A_1 and A_2 equals row i of A. Then $D(A)$ satisfies property 3a.