Cofactors and Consequences (HW #6)

This material is in Section 5.2 of the 5th edition of your textbook.

There are other ways to calculate determinants; we will discuss one of them that leads to applications.

Provide properties that explain each step in the following calculation.

$$\begin{vmatrix} 4 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$$

$$= 4(5 \cdot 4 - 2 \cdot 1) - 2(1 \cdot 4 - 3 \cdot 1) + 3(1 \cdot 2 - 3 \cdot 5)$$

$$= 4 \begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} + 2(-1) \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= (4, 2, 3) \cdot \left(\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix}, (-1) \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} \right)$$

$= \text{(Row 1) (Row 1 of the cofactor matrix)}.$

This is called a cofactor expansion of the first row. Each entry of the first row of $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 2 & 4 \end{bmatrix}$ is in red in the above derivation and the red entry's corresponding **cofactor** is in black next to it. The cofactor consists of a plus or minus sign and a determinant found by eliminating the row and column that the entry is in. If A is an n x n matrix, let A_{-ij} equal the matrix A with row i and column j removed. Then

$$C_{ij} = (-1)^{i+j} \det(A_{-ij}) \text{is the ij-th cofactor of } A$$

and the matrix C is the cofactor matrix of A.
We have just shown that $\det(A) =$ (row 1 A)$\cdot$ (row 1 C). We can show in a similar fashion that

Theorem: Cofactor Theorem

$$\begin{cases} \text{(Row i } C) \cdot \text{(Row j } A) \\ \text{(Col i } C) \cdot \text{(Col j } A) \end{cases} = \begin{cases} \det(A) & i = j \\ 0 & i \neq j \end{cases}. \qquad \text{More briefly, } C^T A = \det(A) I = C A^T.$$

Find a row and column of the cofactor matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 2 & 4 \end{bmatrix}$ and then verify the theorem.

Why is the orthogonal part of the theorem true? Consider how the columns and rows of C are made. For instance, (Col 3 C) does not use any numbers from (Col 3 A). (Col 3 A) could be anything, and yet (Col 3 C) would not change. When dotting (Col 3 C) with (Col 2 A), then, what matrix am I finding the determinant of?

Calculate the determinants of the following matrices.

$$M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Solve $\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$ for λ.

Solve $|K - \lambda I| = 0$ if $K = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.