Determinant Applications (HW #6)

This material is in Section 5.3 of the 5th edition of your textbook.

Application 1: The Cofactor Theorem implies $C^T A = \det(A) I$ when C is the cofactor for A. If A is invertible, then multilying on the right on each side by A^{-1} and dividing by $\det(A)$ gives us

$$A^{-1} = \frac{C^T}{\det(A)}$$

Find $(A^{-1})_{32}$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 2 & 4 \end{bmatrix}$. Use our new formula.

Application 2: Cramer's Rule

Now let $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $A\vec{x} = \vec{b}$. If A is invertible, then

$$\vec{x} = A^{-1}\vec{b} = \frac{C^T\vec{b}}{\det(A)}.$$

For example, solve $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$ if $\vec{x} = (x_1, x_2, x_3)$ using the formula above.

Cramer's Rule:

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $A\vec{x} = \vec{b}$, then $x_i = \dfrac{\det(B)}{\det(A)}$ where $B = A$ with (col i A) changed to $\vec{b}$.

Variation of parameters uses this rule in the 2 x 2 case in most differential equations classes. You might be solving (without knowing it)

$$\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and end up with $v_1' = \dfrac{2xe^{2x} - e^{2x}}{e^{3x}}$ and $v_2' = \dfrac{e^x - xe^x}{e^{3x}}$ from Cramer's Rule. Write v_1' and v_2' using determinants.

Application 3: Cross Product

In math 160 and math 200 you were told the cross product of two vectors $\vec{a} \times \vec{b}$ was a vector perpendicular to and had norm equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$. Then I hope they derived a formula for finding the cross product that uses determinants:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This was really notation saying the cross product was equal to the first row of the cofactor matrix. The cofactor theorem then gives all the properties of the cross product.

We can now generalize the cross product. Put n-1 vectors with n components in rows 2 through n of an n x n matrix. It doesn't matter what the first row is, so put the standard basis for $\mathbb{R}^n$ there: $\hat{e}_1, \hat{e}_2, ..., \hat{e}_n$. Then the cross product of the n - 1 vectors is the first row of the cofactor matrix.

Find the cross product of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Application 4: Volume = |Determinant|

I'm also hoping you were told the **absolute value** of the triple product $\vec{a} \times \vec{b} \cdot \vec{c}$ equals the volume of the box spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$. But since the cross product is just the first row of the cofactor matrix, this is equal to the absolute value of the determinant $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

We can generalize this notion to any number of dimensions.

What is the volume of the four dimensional box spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \end{bmatrix}$?

This page is only for those who like proofs. I will not test you on this material.

The proof of application 4 is similar to the proof of the product rule of determinants. We show that $\pm$ Volume of the box spanned by n vectors satisfies the first three properties of determinants and so must be, by definition, the absolute value of the determinant of those n vectors.

Property 1: The rows of I span a unit cube which has volume equal to 1.

Property 2: If two rows are switched then the volume stays the same, so $\pm$ volume satisfies property two.

Property 3a and 3b depend on the formula Volume = (Base)(Height) where Height is a one dimensional quantity and Base is then a n-1 dimensional quantity: for example, a three dimensional box could have a Height measured in meters and Base measured in square meters. We can decide to make the height in any direction perpendicular to any face of the box.

Property 3a: If one row is multiplied by a constant k, the Height in that direction is multiplied by k so the volume increases by a factor of k, satisfying property 3a.

Property 3b: If one row $\vec{R}$ is the sum of two vectors $\vec{R_1}$ and $\vec{R_2}$, then the height H of $\vec{R}$ in that direction is the sum of the heights of $\vec{R_1}$ and $\vec{R_2}$, $H_1 + H_2$. Each of the component heights is multiplied by the same Base to get the volume. But then Volume = (Base)H=(Base)(H_1+H_2) = (Base)H_1 + (Base)H_2 is the sum of the component volumes which implies Property 3b is satisfied.

As usual, property 3b is hardest to understand; I hope the schematic below helps. Remember that the Base is an n-1 dimensional face of the box, but the height is one dimensional.