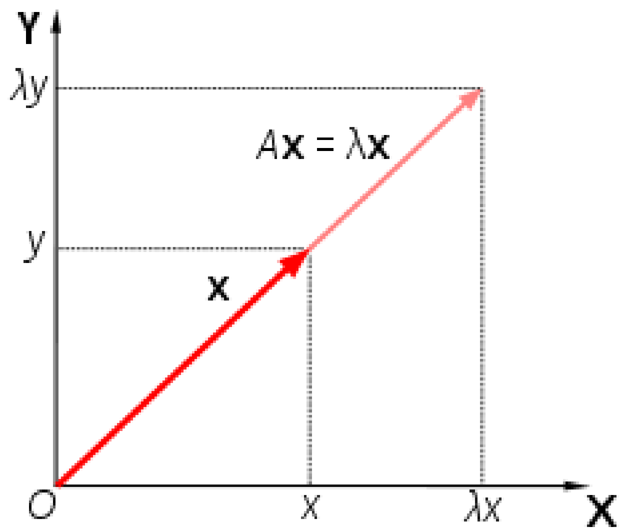
Eigenvalues and Eigenvectors (HW #6)

This material is in Section 6.1 of the 5th edition of your textbook.

λ is an **eigenvalue** of an n x n matrix A with corresponding **eigenvector** $\vec{x}$ if and only if $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda\vec{x}$.

Geometrically the product of $A\vec{x}$ is either $\vec{0}$ or a nonzero vector parallel to A.



$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, so this matrix reflects vectors across the line $y = x$. Find the eigenvectors and corresponding eigenvalues by referring to a picture of reflected vectors.

How do we find eigenvalues and eigenvectors? Usually numerically since the following theory shows an n x n matrix will have n eigenvalues (counting repetitions and complex values) that can only be found by solving a degree n polynomial. However, the method always works for n =2.

$$\begin{aligned} A\vec{x} = \lambda\vec{x} &\implies A\vec{x} - \lambda\vec{x} = \vec{0} \\ &\implies (A - \lambda I)\vec{x} = \vec{0} \\ &\implies \vec{x} \in N(A - \lambda I). \end{aligned}$$

But an eigenvector is nonzero, so if such a vector exists, then $A - \lambda I$ must be singular so that

$$|A - \lambda I| = 0$$

has a solution.

Our tactics, then are to

1) Solve $|A - \lambda I| = 0$ for λ to find all possible eigenvalues.

2) Back substitute λ into $A - \lambda I$ and find its nullspace. $N(A - \lambda I)$ is often called the **eigenspace** for λ.

Find the eigenvalues and eigenspaces for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Notes:

1) The trace of A is the sum of its diagonal entries. Notice in this case that the trace of A equals the sum of the eigenvalues and $\det(A)$ is the product of the eigenvalues. This is true for any n x n matrix A!! For the 2 x 2 case, if we set $\tau =$ trace of A and $\Delta = \det(A)$, then

$$\lambda^2 - \tau\lambda + \Delta = 0$$

so that the quadratic equation gives

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

.

2)Since the dimensions of the eigenspaces add up to 2, we say that A is **complete** or **has no defect**. In general, an n x n matrix is complete if the dimensions of its eigenspaces add to n and defective if that is not the case.

Find the eigenvalues and eigenspaces for $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Find the eigenvalues and eigenspaces for $K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

It is always true that $N(A - \lambda I) = \overline{N\left(A - \bar{\lambda} I\right)}$, so you may conjugate and skip the elimination work for the second eigenspace.

For instance, if $\lambda = 5 + 6i$, and $N(A - \lambda I) = c \begin{bmatrix} -2 + 3i \\ 3 - 4i \end{bmatrix}$ then what is $N(A - \bar{\lambda} I)$?

Find the eigenvalues and eigenspaces for $K = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Notice some shortcuts like singularity and easy column after using the standard procedure.

Find the eigenvalues and eigenspaces for $K = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Notice some shortcuts like shifting and sums of rows.

The last matrix is very nice: it has only real eigenvalues and is complete. Notice the matrix is symmetric. This will always be true for symmetric matrices - it is a famous fact called the Spectral Theorem. That theorem says one more thing will occur with regard to the eigenspaces. Can you guess what it is?

Answer true or false and defend your answer.

1) A and A^T have the same eigenvalues.

2) A and A^T have the same eigenvectors.

3) A, $A + 3I$, $4A$, A^2, and A^{-1} have the same eigenvectors.

How do the eigenvalues of the matrices in problem 3) change?