Diagonalization (HW #7)

This material is in Section 6.2 of the 5th edition of your textbook.

Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are **independent** eigenvectors with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively for a n x n matrix A.

Let X be the n x n matrix with (col i X)=$\vec{x}_i$. Then A(col i X)=$\vec{x}_i\lambda_i$ implies

$$AX = X\Lambda$$

where Λ is the n x n diagonal matrix with $\Lambda_{ii} = \lambda_i$. Because the $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are **independent** X is invertible, so we have a new factorization of A:

$$A = X\Lambda X^{-1}.$$

When this occurs, we say A can be **diagonalized** since we could have solved for Λ and gotten $\Lambda = X^{-1}AX$.

Factor $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ into $X\Lambda X^{-1}$ form if possible. Otherwise explain why it is impossible to do so.

Factor $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ into $X\Lambda X^{-1}$ form if possible. Otherwise explain why it is impossible to do so.

Factor $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ into $X\Lambda X^{-1}$ form if possible. Otherwise explain why it is impossible to do so.

When a matrix does not have enough independent eigenvectors to be diagonalized, we say that matrix is **incomplete** or **defective**. This can only happen if a matrix has a repeated eigenvalue - however, there are plenty of diagonalizable matrices with repeated eigenvalues (I for instance.)

If A is diagonalizable, then A^n is easy to calculate since the inner XX^{-1}'s equal I.

$$A^n = X\Lambda^n X^{-1}$$

Using work from our first example, calculate A^{20} if $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

Using work the work above, write the rank-1 decomposition of the diagonalization of $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

Recall the taylor series for e^x expanded about $a = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use the taylor series to define e^A and then calculate it if $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

This idea that $A = X\Lambda X^{-1}$ has been generalized into the notion of **similarity**. A n x n matrix A is **similar** to a n x n matrix B if there is an invertible matrix X so that $A = XBX^{-1}$.

True or False: If A is similar to B, then the eigenvalues of A equal the eigenvalues of B. This is clearly false if "eigenvalues" is replaced by "eigenvectors" since the eigenvectors of Λ for $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ are $N(\Lambda - I) = c(1, 0)$ and $N(\Lambda - 2I) = c(0, 1)$ which are different from $N(A - I) = c(2, 1)$.

True or False: If A is similar to B, then $\det(A) = \det(B)$.

Theorem: If a set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of a n x n matrix A are all distinct (all different,) then any set of corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly independent.

The proof is only for those interested and if we have time; I won't ask you to repeat it.

If $k = 2$, and $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$, then multiplying both sides by $(A - \lambda_1 I)$ gives

$$(A - \lambda_1 I)(c_1\vec{x}_1 + c_2\vec{x}_2) = c_2(A - \lambda_1 I)\vec{x}_2 = \vec{0}.$$

Since $\vec{x}_2$ is not an eigenvector of λ_1, the "distinct" hypothesis guarantees $(A - \lambda_1 I)(\vec{x}_2) \neq \vec{0}$ so that

$$c_2 = 0.$$

$\vec{x}_1 \neq \vec{0}$ since it is an eigenvector so $c_1\vec{x}_1 = \vec{0}$ implies

$$c_1 = 0.$$

Therefore the two vectors must be independent.

We can prove the case where k is larger than two in a similar fashion by repeatedly multiplying by $(A - \lambda_i I)$ to show only the trivial linear combination equals zero.

If the class is interested and there is time, I will prove the case for $k = 3$ and/or use induction to prove true for any k.