Markov Matrices (HW #7)

This material is in Section 10.3 of the 5th edition of your textbook.

M is a **Markov Matrix** if each of its entries $M_{ij} \geq 0$ and the column entries add to 1; this second condition can be written briefly as $M^T\vec{1} = \vec{1}$.

But then 1 is an eigenvalue of M^T and so is also one for M since (as we have proved already) transposes have the same eigenvalues as the original matrix.

A second important consequence is

$$\vec{1}^T M\vec{v} = \vec{1}^T\vec{v}$$

so that **the sum of the components of a vector is preserved by multiplication by a Markov Matrix!** My shorthand language for this is: "Markov matrices preserve the number of objects".

Finally, we can also prove that the magnitude of any eigenvalue in M must be less than or equal to one[1].

Give me two 2 x 2 examples of Markov Matrices and find their eigenvalues. Multiply each of them by $(2, 5)$ and notice how we always get a new vector with sum of components equal to 7.

This is a good time to review rank-1 decomposition of multiplication. What is the rank one decomposition of $\begin{bmatrix}1 & 2 & 3\\4 & 1 & 4\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\0 & 5 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}3 & 1 & 2\\1 & 1 & 6\\4 & 3 & 1\end{bmatrix}$?

[1]The proof shows that all eigenvalues must be in the **Gershgorin Discs** of the columns of M which are centered about the M_{ii} with radius smaller than $\sum_{j\neq i}|M_{ij}|$. Since the columns add to one, the discs must have upper bound less than or equal to 1 and lower bound greater than or equal to -1.

Suppose when I was 23 every month the number of my acquaintances remained the same, but might change categories between "friends," "lovers," or "enemies." Let a vector

$$\vec{u}_n = \begin{bmatrix} \#\text{ friends} \\ \#\text{ lovers} \\ \#\text{ enemies} \end{bmatrix}$$

represent these numbers at the end of month n. After plenty of research, a "transition matrix" is constructed with entries equal to probabilities

$$M = \begin{bmatrix} .5 & .5 & .5 \\ .3 & .4 & 0 \\ .2 & .1 & .5 \end{bmatrix}$$

where M_{ij} equals the probability that a person in category j will change to category i. For instance, if

$$\vec{u}_0 = \begin{bmatrix} 24 \\ 1 \\ 3 \end{bmatrix}$$

then

$$\vec{u}_1 = \begin{bmatrix} .5 & .5 & .5 \\ .3 & .4 & 0 \\ .2 & .1 & .5 \end{bmatrix} \begin{bmatrix} 24 \\ 1 \\ 3 \end{bmatrix}$$

expressed as a linear combination of columns is

$$= 24 \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} + 1 \begin{bmatrix} .5 \\ .4 \\ .1 \end{bmatrix} + 3 \begin{bmatrix} .5 \\ 0 \\ .5 \end{bmatrix}$$

so, for instance, half my friends remained friendly, .3 became lovers, and .2 became enemies, but the new state has the same number of aquaintances - the number of objects is preserved when multiplying by a Markov matrix:

$$= \begin{bmatrix} 14 \\ 7.6 \\ 6.4 \end{bmatrix}$$

The question is: if my transition matrix does not change (so my behavior remains the same) what will the distribution be after many months? That is, find $\vec{u}_\infty$. If such a vector exists, it is called the **steady state** vector.

We can answer this question by beginning to diagonalize M.

Begin diagonalizing M. Stop after finding the eigenvalues and two eigenspaces.

$\vec{u}_n = M^n \vec{u}_0$. What is M^n and what happens as $n \to \infty$? Use a rank-1 decompostion of $X \Lambda X^{-1}$ to determine $\vec{u}_\infty = \lim\limits_{n \to \infty} M^n \vec{u}_0$. Remember the number of acquaintances is preserved at 28.

Not all Markov matrices have a steady state vector. For instance, $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has an "orbit," not a steady state. Suppose our vectors only have two columns, friends and enemies. What does this Markov matrix do to the initial vector $\vec{u}_0 = \begin{bmatrix} 24 \\ 4 \end{bmatrix}$?