Systems of Differential Equations (HW #7)

This material is in Section 6.3 of the 5th edition of your textbook.

A differential equation is an equation with derivatives of a function and the solution is that function. I hope you learned about the simple equation $y' = ky$ in second semester calculus where k is a constant: the general solution can be found by guessing and checking: $y = Ce^{kx}$. If an intial condition is given, $y(0) = y_0$, then we can solve for C and get the particular solution.

Equations with higher order derivatives are useful too, and it is an interesting fact that we can represent any such equation as a system of first order equations:

$$\vec{u}\,' = A\vec{u}.$$

In this class we will solve this system when A is a complete matrix with real entries and real eigenvalues. Complex eigenvalues and defective matrices are covered in math 220.

By guessing and checking as we did in math 160, we observe that

$$(e^{\lambda t}\vec{v})' = \lambda e^{\lambda t}\vec{v}$$

if $\vec{v}$ is constant, and, if $\vec{v}$ is an eigenvector of A, then

$$A(e^{\lambda t}\vec{v}) = \lambda e^{\lambda t}\vec{v}$$

too.

Thus $e^{\lambda t}\vec{v}$ is a solution to $\vec{u}\,' = A\vec{u}$ if $\vec{v}$ is an eigenvector of A.

A theorem from the study of differential equations called "Existence-Uniqueness" can be used to show the solution set of $\vec{u}\,' = A\vec{u}$ with n x n A will be an n-dimensional vector space[1]; hence if we find n independent eigenvectors of A, the corresponding solutions will form a basis for all solutions.

Our tactics, then, for solving these systems of differential equations will be to find n linearly independent eigenvectors and take the corresponding span for our general solution.

[1] The solution set is indeed a null space for the linear operator $D - I$ where D represents differentiation: $(D - A)\vec{u} = \vec{0}$ is an equivalent system to $\vec{u}\,' = A\vec{u}$

$\begin{cases} x(t) = \# \text{ prey} \\ y(t) = \# \text{ predators} \end{cases}$. You get to choose the prey and predators[2]. After research we found the system governing this interaction was $\begin{cases} x'(t) = 4x - 2y \\ y'(t) = x + y \end{cases}$. If we let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{u}(0) = \begin{bmatrix} 100 \\ 5 \end{bmatrix}$ find $\vec{u}$. What will the ratio of prey to predators be in the long run?

[2]My favorite class choice was cobras as prey and honeybadgers as predators.

By the way, since $y' = ky \implies y = Ce^{kx}$, it seems reasonable that $\vec{u}\,' = A\vec{u} \implies \vec{u} = e^{A}\vec{C}$. Let's see if $e^{A}\vec{C}$ is equal to the solution for our predator/prey problem above.

Sketch the graph of the "phase plane" for the general solution to the predator/prey problem: this is the graph in the xy-plane as t varies.