Spectral Theorem (HW #7)

This material is in Section 6.4 of the 5th edition of your textbook.

This section continues our study of A^TA. We have already seen how A^TA helps find projections and how A^TCA is used to solve a circuit. Both of these matrices are symmetric and this section is about symmetric matrices.

The Spectral Theorem says that if A is an n x n symmetric matrix with real entries, then
1) the eigenvalues of A are real;
2) the eigenspaces are orthogonal to each other;
3) A is complete: if λ is a multiple eigenvalue k times, then the corresponding eigenspace has dimension k; A has no defect.

Diagonalize $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Use an orthonormal basis for X. You might have to use Gram-Schmidt for the repeated eigenvalues.

Now use the diagonalization above and write $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ in rank-1 form $A = \sum_{i=1}^{n} \lambda_i \vec{q}_i \vec{q}_i^T$. Each term in this form represents one "color" of a "rainbow" and makes a spectrum for A. We tend to order the eigenvalues from largest to smallest (though some applications prefer ascending order instead.)

Engineers often call the Spectral Theorem the **Principal Axis Theorem** because of this rank-1 decomposition. To understand why, we need to review ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the standard form for an ellipse centered at the origin with principal axes that are the $y = 0$ and $x = 0$ with "half-lengths" equal to a and b respectively.

Find the principal axes and half-lengths of the rotated ellipse $3x^2 - 2xy + 3y^2 = 1$; as a start, notice that $3x^2 - 2xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$; then the square roots of the eigenvalues will be the reciprocals of the half-lengths and the corresponding eigenvectors the principal axes. To see why, diagonalize the symmetric matrix and write the right side in rank-1 form.

Setting $X = \left(\frac{x-y}{\sqrt{2}}\right)$ and $Y = \left(\frac{x+y}{\sqrt{2}}\right)$ then allows us to rewrite $3x^2 - 2xy + 3y^2 = 1$ as $4X^2 + 2Y^2 = 1$. Setting $Y = 0$ and $X = 0$ gives the principal axes and square roots of $1/2$ and $1/4$ give the half-lengths.

This **change of basis** allows us to solve the problem easily.

Recall The Spectral Theorem says that if A is an n x n symmetric matrix with real entries, then
1) the eigenvalues of A are real;
2) the eigenspaces are orthogonal to each other;
3) A is complete.

We can prove 1) and 2). The proof for 3) goes beyond the scope of this class.

Recall that if $z = a + bi$, then the conjugate of z is $\bar{z} = a - bi$, and that $z\bar{z} = a^2 + b^2$ is a real number. Also notice that $\bar{\bar{z}} = z$. It is also easy to prove $\overline{zw} = \bar{z}\bar{w}$.

1) If A is symmetric and $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$, then multiplying both sides by $\bar{\vec{x}}^T$ gives

$$\bar{\vec{x}}^T A\vec{x} = \lambda\bar{\vec{x}}^T\vec{x}.$$

Since $\bar{\vec{x}}^T\vec{x}$ is a real number, λ will be real if the 1 x 1 matrix product $\bar{\vec{x}}^T A\vec{x}$ is real. That will be the case if it is equal to its conjugate. Since a 1 x 1 matrix product equals its transpose, we will be done if the conjugate transpose of $\bar{\vec{x}}^T A\vec{x}$ equals $\bar{\vec{x}}^T A\vec{x}$. Show this is true.

2) Suppose $\vec{x}$ and $\vec{y}$ are eigenvectors of A with distinct eigenvalues $\lambda_x \neq \lambda_y$ respectively. Then $\vec{x} \in N(A - \lambda_x I)$, but $\vec{y}$ is not. In fact,

$$(A - \lambda_x I)\vec{y} = A\vec{y} - \lambda_x\vec{y} = (\lambda_y - \lambda_x)\vec{y} \implies \vec{y} \in C(A - \lambda_x I).$$

Since $A - \lambda_x I$ is symmetric, $\vec{y}$ must also be in its row space and hence orthogonal to its null space by the FTLA. Hence $\vec{x} \cdot \vec{y} = 0$ and we are done with the proof.