Positive Definite Matrices (HW #8)

This material is in Section 6.5 of the 5th edition of your textbook.

If M is any m x n matrix with **independent columns** then we can prove $M^T M$ is **positive definite**. I better define what that means before I make any more statements about it!

Every n x n symmetric matrix A with real entries has a corresponding quadratic polynomial $\vec{x}^T A\vec{x}$.

If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $A = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix} \implies \vec{x}^T A\vec{x} =$

If $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then $B = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} \implies \vec{x}^T B\vec{x} =$

Find $\vec{x}$ and A so that $10x^2 + 5y^2 - 8z^2 + 2xy - 4xz + 6yz = \vec{x}^T A\vec{x}$.

When does $f(\vec{x}) = \vec{x}^T A\vec{x}$ have an isolated minimum of 0? When A is positive definite.

A symmetric matrix A is **positive definite** if and only if $\vec{x}^T A\vec{x} > 0$ when $\vec{x} \neq \vec{0}$.

How can we tell if a symmetric matrix is positive definite? We need a theorem for that.

Positive Definite Equivalency Theorem - If A is an n x n symmetric matrix, then the following statements are equivalent.

1) $\vec{x}^T A\vec{x} > 0$ when $\vec{x} \neq \vec{0}$. (That is, A is positive definite.)

2) The n eigenvalues, counting multiple ones, of A are all positive.

3) A has n positive pivots.

4) LPMD's > 0. (upper Left leading Principal Minor Determinants are all positive.

5) There is a matrix C with independent columns so that $A = C^T C$. This is called the "Cholesky factorization."

Is $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ positive definite?

Is $B = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$ positive definite?

Is $C = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ positive definite?

Is $D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ positive definite?

True or False? If the columns of m x n A are independent and the entries of A are real, then $A^T A$ is positive definite. Hint: the definition is best of the equivalencies in this case.

Proof only if time and desire allows.

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1) $\vec{x}^T A\vec{x} > 0$ when $\vec{x} \neq \vec{0}$.
2) The n eigenvalues of A are all positive.
3) A has n positive pivots.
4) LPMD's > 0.
5) There is a matrix C with independent columns so that $A = C^T C$.

proof: 5) $\implies$ 1): $\vec{x}^T A\vec{x} = (\vec{x}^T C^T)(C\vec{x}) = \|C\vec{x}\|^2 \geq 0$. Since columns of C are independent, then $C\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$, proving 1).

1) $\implies$ 2): $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$, implies $\lambda = \dfrac{\vec{x}^T A\vec{x}}{\|\vec{x}\|^2} > 0$ proving 2).

2) $\implies$ 5): $A = Q^T \Lambda Q = (\sqrt{\Lambda}Q)^T(\sqrt{\Lambda}Q)$ where $\sqrt{\Lambda}_{ij} = \begin{cases} 0 & i \neq j \\ \sqrt{\lambda_i} & i = j \end{cases}$ is defined because all eigenvalues are positive. The columns of $C = \sqrt{\Lambda}Q$ are independent since $\sqrt{\Lambda}Q$ is invertible.

So statements 1, 2, and 5 are all equivalent.

3) $\implies$ 4): the ith LPMD equals the product of the first i pivots which are all larger than zero.

4) $\implies$ 3): first pivot = first LPMD > 0. By induction, if the first k pivots are all positive then the k + 1 pivot is too since the positive (k+1)st LPMD equals the product of first k pivots (all positive) and the (k+1) pivot.

so statements 3 and 4 are equivalent.

3) $\implies$ 5): $A = U^T DU$ is the LDU factorization since A is symmetric. Then $A = (\sqrt{D}U)^T(\sqrt{D}U)$.

Now I just need to prove 1) $\implies$ 3) to prove equivalency because 1) is equivalent to 5).
$A = LDL^T$, so $0 < \vec{x}^T A\vec{x} = \vec{x}^T LDL^T\vec{x} = \displaystyle\sum_{i=1}^{n} d_i(\vec{l_i} \cdot \vec{x})^2$ for any nonzero $\vec{x}$. Let $\vec{x}_j$ be a nonzero vector orthogonal to all but the jth column of L, $\vec{l_j}$. Then

$$0 < \vec{x}_j^T LDL^T\vec{x}_j = \sum_{i=1}^{n} d_i(\vec{l_i} \cdot \vec{x}_j)^2 = d_j(\vec{l_j} \cdot \vec{x}_j)^2$$

which can only happen if $d_j > 0$. Since this works for any j, all pivots must be positive.