Linear Transformations (HW #9)

If V and W are vector spaces, then a function with domain V and range inside W is a **linear transformation** if:

1) $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$
and
2) $L(c\vec{x}) = cL(\vec{x})$.

1) in english is "L distributes over vector addition" or **DOVA** (not to be confused with CUVA)
and
2) in english is "L commutes over scalar multiplication" or **COSM** (not to be confused with CUSM.)

Common notation is $L : V \to W$.

Multiplication by a matrix A on the left DOVA and COSM, so if A is an m x n matrix, then the function $L(\vec{x}) = A\vec{x}$ is a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$. The FTLA tells us all about this transformation, that the range is $C(A)$ and the vectors sent to $\vec{0}$ are in $N(A)$.

If there is a m x n matrix $[L]$ for which $L(\vec{x}) = [L]\vec{x}$, we say that $[L]$ is the matrix of the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$. $[L]$ always exists and can be found column by column since

$$L(\text{Col i I}) = [L](\text{Col i I}) = \text{Col i [L]}.$$

Every type of matrix has a corresponding linear transformation: elementary matrices, projection matrices, diagonal matrices, and orthogonal matrices. In particular, orthogonal matrices are rotation or reflection matrices and diagonal matrices contort the dimensions, so the SVD says each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the composition of a rotation/reflection with a contortion and with another rotation/reflection. Circles are thus sent to ellipses. Look at the matlab notes for homework 9 for pictures of this.

Find the matrix for the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ that first rotates 60 degrees C.C.W. about the z-axis, then 90 degrees C.C.W. about the x-axis before reflecting about the xy-plane. Hint: Multiply three matrices that correspond to each separate motion.

Here is language used for linear transformations:

1) the **kernel** of L equals all vectors for which $L(\vec{x}) = \vec{0}$. The kernel of $L = N([L])$ when $[L]$ exists.

2) the **range** of L is the set of all outputs and is equal to $C([L])$ when $[L]$ exists.

Find the kernel and range of $g(x, y, z) = (3x + y - z, 2x - 5y)$.

Let D be differentiation, $D : \mathbb{P}_3 \to \mathbb{P}_2$ where $\mathbb{P}_n$ = polynomials of degree $\leq$ n. Show that D is a linear transformation.
What is $[D]$? You have to find a basis for P_n first and declare it to be the standard basis.

Is $T : \mathbb{R}^2 \to \mathbb{R}^2$ a linear transformation if $T(\vec{v}) =$ the rotation of $\vec{v}$ about the origin by θ radians? Defend your answer.

$L : V \to W$ is **one-to-one** if $L(\vec{x}) = L(\vec{y})$ if and only if $\vec{x} = \vec{y}$. L is **onto** if $L(V) = W$. Using the last two examples D and T, are either one-to-one? Are either of them onto?

AS TIME ALLOWS

Let $V = \text{span}\{\cos(x), \cos(2x), \sin(x), \sin(2x)\}$. Assume the standard basis is the spanning set, and again assume D is differentiation. What is $[D]$?

Is $P : \mathbb{R}^3 \to \mathbb{R}^3$ a linear transformation if $P(\vec{v})$ = the projection of $\vec{v}$ onto $(1, 1, 1)$? Defend your answer. Is P one-to-one or onto?