20 points   Math 220   Homework 1   Sections 0.1-0.3, 1.1-1.4   18 problems

Most of these problems are from Jiri Lebl's "Notes on Diffy Q's, Differential Equations for Engineers."

1. Solve the following equations by guess and check.

   A) $\frac{dA}{dt} = -10A$, $A(0) = 5$ B) $\frac{d^2x}{dt^2} = -9x$, $x(0) = 1$, $x'(0) = 0$.

2. Find all b so that the Existence-Uniqueness theorem guarantees a unique local solution for $y' = \cos(t)\sqrt{1-y^2}$ if $y(0) = b$. Defend your answer.

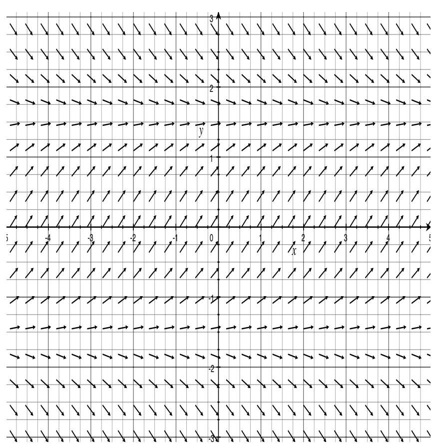
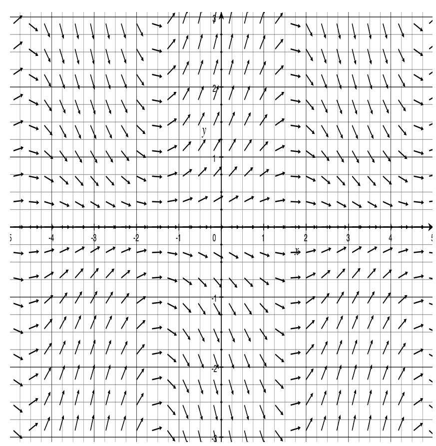
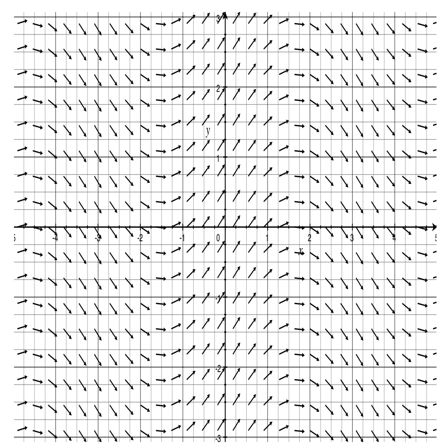
3. The population of a town grows at a rate proportional to the population present at time t. The population increases by 15% in 10 years. What will be the population in 30 years if the initial population is 500? How fast is the population growing at t = 30 years? Set up and solve a differential equation to find your answers.

4. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay? Set up and solve a differential equation to find your answers.

5. Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique? Answer "yes" or "no" and defend.

6. Is it possible to find a unique solution for $y' = \frac{x}{x^2 - 1}$ if $y(1) = 0$? Answer "yes" or "no" and defend.

7. Match equations $y' = \cos(x)$, $y' = \cos(y)$, $y' = y\cos(x)$ to slope fields. Justify your answers.

a)  b)  c) 

8. Solve $y' = 2xy$.

9. Solve $x' = 3xt^2 - 3t^2$ if $x(0) = 2$.

10. Find an implicit solution for $x' = \frac{1}{3x^2 + 1}$ if $x(0) = 1$.

11. Find an explicit solution for $xy' = y^2$ if $y(1) = 1$.

12. Find an implicit solution for $y' = \frac{\sin(x)}{\cos(y)}$.

13. Solve $y' + 3x^2y = 2xe^{-x^3}$.

14. Solve $y' + 2\sin(2x)y = 2\sin(2x)$ if $y(\pi/2) = 3$.

15. Suppose a water tank is being pumped out at 3 L/min. The water tank initially has 10 L of clean water. Water with toxic substance is flowing into the tank at 2 L/min, with concentration $20t$ g/L at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

16. A toxic substance is slowly being added to bacteria on a plate slowing down the rate of growth. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.

17. Solve $(x+1)\frac{dy}{dx} + y = \ln(x)$ if $y(1) = 10$ and give the interval on which the solution is defined.

18. CAS problem (3 points): use a CAS device for the following. Turn in a pdf file as specified in the syllabus. **Matlab notes for this lab follow the brief answers.**

    A) Find the exact solution for the IVP $y' = -100y + 100\sin(t)$ if $y(0) = 0$. Approximate $y(3)$ to four digits.
    B) Find the exact solution for the IVP $y' = -100y + 100\sin(t)$ if $y(0) = 3$. Approximate $y(3)$ to four digits.
    C) Draw the slope field for $y' = -100y + 100\sin(t)$. Use the slope field to help explain why the two approximations for $y(3)$ from part A) and B) are the same.

Brief answers

1. a) $A = 5e^{-10t}$ b) $x = \cos(3t)$
2. $-1 < b < 1$
3. $P(30) = 500(1.15)^3 \approx 760$ people, and $P'(30) \approx 76.0\ln(1.15) \approx 10.6$ people / year.
4. 11 hours
5. Yes, a solution exists (and is unique.)
6. No. If we assume the solution is differentiable at $x = 1$, no solution exists. Otherwise there are many possible solutions.
7. a) $y' = \cos y$ b) $y' = y\cos(x)$ c) $y' = \cos x$
8. $y = ce^{x^2}$
9. $x = e^{t^3} + 1$
10. $x^3 + x = t + 2$
11. $y = \dfrac{1}{1 - \ln(x)}$
12. $\sin(y) = -\cos(x) + C$
13. $y = e^{-x^3}(x^2 + C)$
14. $y = 2e^{\cos(2x)+1} + 1$
15. 250 grams
16. 6.31×10^6
17. $y = \dfrac{x\ln(x) - x + 21}{x + 1}$

MatLab Notes

We use symbolic variables in MatLab when using the dsolve command. Below I obtain a particular solution for $y' = 2y$ if $y(0) = 5$ and then evaluate the solution when $x = 3$; the vpa command then gives a six digit estimate for that solution. Then I find the general solution for $y'' = -9y$.
Variables are numeric unless declared to be symbolic. Numeric variables are vectors, so if we want to perform arithmetic component by component, we must use a dot before multiplication, division, or exponentiation commands. Addition and subtraction are done component by component in vectors, so we still use + or − without a period in front. Numeric variables, X and Y, are used below to make the slope field. Notice the capitals are different variables from the lower case symbolic variables x and y.

```
syms y(x)
ysol(x) = dsolve(diff(y,x)==2*y, y(0)==5)
```

ysol(x) = $5\,e^{2x}$

```
ysol(3), vpa(ysol(3),6)
```

ans = $5\,e^{6}$

ans = 2017.14

```
dsolve(diff(y,x,x)==-9*y)
```

ans = $C_6 \cos(3\,x) - C_7 \sin(3\,x)$

```
[X,Y]=meshgrid([-4:0.2:4],[-4:0.2:4]);
U=ones(size(X));
V=Y.^2-X;
L=sqrt(1+V.^2);
quiver(X,Y,U./L,V./L,0.5);
axis tight
```