20 points   Math 220   Homework 3   Sections 2.4 - 2.6   18 problems

Most of these problems are from Jiri Lebl's "Notes on Diffy Q's, Differential Equations for Engineers."

We use the mks units (meters-kilograms-seconds) in these homework problems unless stated otherwise.

1. A spring-mass system has a mass m = 2, spring constant k = 3, and damping constant c = 1.
   a) Set up and find the general solution of the system.
   b) Is the system under damped, over damped or critically damped?
   c) If the system is not critically damped, find a c that makes the system critically damped.

2. A spring-mass system has a mass m = 3, spring constant k = 12, and damping constant c = 12.
   a) Set up and find the general solution of the system.
   b) Is the system under damped, over damped or critically damped?

3. A spring with spring constant 4 N/m. You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion.
   a) You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?
   b) Find a formula for the mass m given the frequency f in Hz.

4. A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time t = 0 the mass is kicked and starts traveling at 2 meters per second. How large does k have to be so that the mass does not go further than 3 meters from the rest position?

5. Suppose we have an RLC circuit with a resistor of 100 milliohms (0.1 ohms), inductor of 50 millihenries (0.05 henries), and a capacitor of 5 farads, with constant voltage.
   a) Set up the ODE equation for the current I.
   b) Find the general solution. Write your final answer as a single term with a cosine but no sine.
   c) Find a particular solution so that $I(0) = 10$ and $I'(0) = 0$.

6. A 5000 kg railcar hits a bumper (a spring) at 1 m/s, and the spring compresses 0.1 meters. Assume no damping.
   a) Find the spring constant k.
   b) Find out how far the spring compresses when a 10000 kg railcar hits the spring at the same speed.
   c) If the spring would break if it compresses further than 0.3 m, what is the maximum mass of a railcar that can hit it at 1 m/s?
   d) What is the maximum mass of a railcar that can hit the spring without breaking at 2 m/s?

7. Find the general solution to $y'' + 3y' - 10y = 6e^{4x}$.

8. Find the general solution to $y'' - 2y' + 5y = 25x^2 + 17$

9. Use the complex analog of the real equation $y'' - y' + y = 2\sin(3x)$ to find a particular solution. You can also find another particular solution from the same work. What is it, and what equation does it solve?

10. Find the general solution to $y'' + 2y = e^x + x^3$. Superposition might help you be more efficient.

11. Solve $y'' - 2y' + y = e^x$ if $y(0) = 1$ and $y'(0) = 2$. Why might you want to use variation of parameters? For this nice problem, either method works well.

12. Use variation of parameters to find a particular solution for $y'' - y = \dfrac{1}{e^x + e^{-x}}$.

    Notice the right side is equal to $\dfrac{1}{2\cosh(x)}$; you might consider a hyperbolic basis for the homogeneous solution.

13. Use variation of parameters to find a particular solution for $y'' + 2y' + 5y = e^{-x}\sec(2x)$.

14. For an arbitrary constant c find the general solution to $y'' - 4y = \sin(x + c)$. Notice there is no y'-term, so $\cos(x + c)$ doesn't have to be in a guess for a particular solution.

In the next three problems, use the equations derived in lecture for resonance. For practical resonance we had

$$mx'' + \alpha x' + kx = F_0 \cos(\omega t)$$

and used $2p = \dfrac{\alpha}{m}$, $\omega_0^2 = \dfrac{k}{m}$, **and** $\tan(\gamma) = \dfrac{2\omega p}{w_0^2 - w^2}$ **then found**

$$x_p = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4p^2\omega^2}} \cos(\omega t - \delta).$$

Then calculus helped us derive the relative frequency that maximizes amplitude:

$$\omega = \sqrt{\omega_0^2 - 2p^2}$$

15. A mass of 4 kg is on a spring with constant $k = 4$ and a damping constant $c = 1$. Suppose the forcing function is $2\cos(\omega t)$. Find the ω that causes practical resonance and find the amplitude.

16. Derive a formula for x_{sp} if $mx'' + \alpha x' + kx = F_0 \cos(\omega t) + A$, where A is a constant. Assume $\alpha > 0$. Use superposition and use the above formula - don't derive it again!

17. Suppose there is no damping in a mass and spring system with $m = 5$, $k = 20$, and $F_0 = 5$. Suppose that ω is chosen to be precisely the resonance frequency.
    a) Find ω. b) Find the amplitude of the oscillations of x_p at time $t = 100$.

18. CAS problem (3 points): use a CAS device for the following. Turn in a pdf file as specified in the syllabus. **Matlab notes for this lab follow the brief answers.**

    Set up and solve the differential equation for the LRC circuit with resistance of 12 ohms, inductance of 0.15 henrys, capacitance of 100 microfarads, an electromotive force of $100\sin(60t)$ volts, initial charge on capacitor of 10 micro coulombs, and initial current of four amperes. The solution is too large to fit on one line, so do not print it. Instead, draw the graph of the solution for $0 \le t \le 0.5$. **Why is the beginning of the graph different from the rest of the graph?**

Brief answers

1. a) $x(t)=e^{-0.25t}\left[C_1\cos\left(\sqrt{23}t/4\right)+C_2\sin\left(\sqrt{23}t/4\right)\right]$ b) under damped c) $c=2\sqrt{6}$

2. a) $x(t)=e^{-2t}(C_1+C_2t)$ b) Critically damped

3. a) $\dfrac{25}{16\pi^2}$ kilograms b) $m=\dfrac{1}{\pi^2 f^2}$

4. $k\geq 8/9$

5. a) $0.05I''+0.1I'+0.2I=0$ b) $I(t)=Ce^{-t}\cos(\sqrt{3}t-\gamma)$ c) $I(t)=\dfrac{20e^{-t}}{\sqrt{3}}\cos(\sqrt{3}t-\pi/6)$

6. a) $k=5\times 10^5$ b) $\dfrac{1}{5\sqrt{2}}\approx 0.141$ c) 45000 kg d) 11250 kg

7. $y=C_1e^{-5x}+C_2e^{2x}+\dfrac{e^{4x}}{3}$

8. $y=e^x\left(C_1\cos(2x)+C_2\sin(2x)\right)+5x^2+4x+3$

9. $y_p=\dfrac{6\cos(3x)-16\sin(3x)}{73}$. The other equation is $y''-y'+y=2\cos(3x)$ and the corresponding particular solution is $y_p[=\dfrac{-6\sin(3x)-16\cos(3x)}{73}$.

10. $y=C_1\cos(\sqrt{2}x)+C_2\sin(\sqrt{2}x)+e^x/3+x^3/2-\dfrac{3}{2}x$

11. $y=e^x(1+x+x^2/2)$. There is overlap.

12. $y_p=\dfrac{x\sinh(x)-\cosh(x)\ln(\cosh(x))}{2}$

13. $y_p=\dfrac{1}{4}\ln(\cos(2x))e^{-x}\cos(2x)+\dfrac{1}{2}xe^{-x}\sin(2x)$

14. $y=C_1e^{2x}+C_2e^{-2x}-\dfrac{\sin(x+c)}{5}$

15. $\omega=\dfrac{\sqrt{31}}{4\sqrt{2}}\approx 0.984$; $C(\omega)=\dfrac{16}{3\sqrt{7}}\approx 2.016$

16. $x_p=\dfrac{F_0}{m\sqrt{(\omega_0^2-\omega^2)^2+4p^2\omega^2}}\cos(\omega t-\delta)+\dfrac{A}{k}$.

17. a) $\omega=2$ b) 25

MatLab Notes start on the next page.

Matlab Notes for Homework #3

A solution to a ciruit without a forcing function.

```
syms Q(t)
Qp(t)=diff(Q,t);
Eqn= 0.05*diff(Q,t,2)+2*diff(Q,t)+100*Q(t)==0;
Cond = [Q(0)==0.1, Qp(0)==0];
q(t)=dsolve(Eqn, Cond)
```

q(t) =

$$\frac{\cos(40\,t)\,e^{-20\,t}}{10} + \frac{\sin(40\,t)\,e^{-20\,t}}{20}$$

Below is a graph of the unforced solution next to the graph of a solution for the same circuit with a nonzero forcing function - a 120 volt AC source - with the same initial conditions.

```
subplot(1,2,1)
fplot(q,[0 0.5])
title('Unforced')
Eqn2=0.05*diff(Q,t,2)+2*diff(Q,t)+100*Q(t)==120*sin(60*t);
q2(t)=dsolve(Eqn2, Cond);
subplot(1,2,2)
fplot(q2,[0 0.5])
title('Forced')
```