20 points Math 220 Homework 6 Sections 1.6, 8.1, 8.2, 4.1, 4.6 18 problems

Most of these problems are from Jiri Lebl's "Notes on Diffy Q's, Differential Equations for Engineers."

1. Let $x' = (x-1)(x-2)x^2$.
   A) Sketch the phase diagram and find the critical points.
   B) Classify the critical points.
   C) If $x(0) = 0.5$ what is $\lim_{t\to\infty} x(t)$? Explain why.

2. Assume that a population of fish, $x(t)$ at time t in a lake initially satisfies $\frac{dx}{dt} = kx(M-x)$ where M is constant. If fish are continually added at A fish per unit of time, what is the new differential equation for $x(t)$? What is the new limiting population?

3. A fish farmer wants to maximize the harvest rate a in tons per week. The population of fish in the absence of harvesting obeys the logistic equation $\frac{dP}{dt} = -P^2 + 10P$.
   a) Sketch the phase portrait and some solutions of the logistic equation. Find and classify each critical point.
   b) Now write the equation with harvest rate a. What harvesting rates will allow for a constant mass of fish?
   c) If the farmer has a constant harvest rate of 16 tons of fish each week over a long period of time, what is the approximate constant mass of fish per week? Defend your answer.

Find and classify (if possible) each critical point as stable or unstable, and as a node, center, spiral, or saddle. Alternatively, you may use "sink" for a stable node and "source" for an unstable node.

4. $\begin{cases} x' = 1 - 2xy \\ y' = 2xy - y \end{cases}$.

5. $\begin{cases} x' = -3x + y^2 + 2 \\ y' = x^2 - y^2 \end{cases}$.

6. $\begin{cases} x' = x(10 - x - 0.5y) \\ y' = y(16 - y - x) \end{cases}$.

7. $\begin{cases} x' = \sin(\pi y) + (x-1)^2 \\ y' = y^2 - y \end{cases}$.

8. The idea of critical points and linearization works in higher dimensions as well with stability determined by the eigenvalues. The types of critical points increase, so we won't worry about those. Find the critical points for $\begin{cases} x' = x + z^2 \\ y' = z^2 - y \\ z' = z + x^2 \end{cases}$. What is the stability of the critical points, and why?

9. Find all eigenvalues and corresponding eigenfunctions for $x'' + \lambda x = 0$, $x(0) = 0$, and $x(\pi) = 0$. Show details.

10. Find all eigenvalues and corresponding eigenfunctions for $x'' + \lambda x = 0$, $x'(0) = 0$, and $x'(1) = 0$. Show details.

11. Find all eigenvalues and corresponding eigenfunctions for $x'' + \lambda x = 0$, $x(0) = 0$, and $x'(\pi) = 1$. Show details.

12. Solve $u_t = u_{xx}$ if $u(0,t) = 0$, $u(1,t) = 0$, and $u(x,0) = \sin(\pi x)$ for $0 \le x \le 1$. Use memorized solutions to the boundary value problems in your work if you can - otherwise derive them.

13. Solve $u_t = u_{xx}$ if $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, and $u(x,0) = 3\cos(x) + \cos(3x)$ for $0 \le x \le \pi$. Use memorized solutions to the boundary value problems in your work if you can - otherwise derive them.

14. Solve $u_t = 3u_{xx}$ if $u(0,t) = 0$, $u(\pi,t) = 0$, and $u(x,0) = 5\sin(x) + 2\sin(5x)$ for $0 \le x \le \pi$. Use memorized solutions to the boundary value problems in your work if you can - otherwise derive them.

15. Solve $u_t = 0.1u_{xx}$ if $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, and $u(x,0) = 1 + 2\cos(x)$ for $0 \le x \le \pi$. Use memorized solutions to the boundary value problems in your work if you can - otherwise derive them.

16. Use separation of variables to find a nontrivial solution to $u_{xt} = u_{xx}$.

17. Use separation of variables to find a nontrivial solution to $u_x + u_t = u$.

18. CAS problem (3 points): use a CAS device for the following. Turn in a pdf file as specified in the syllabus. **No Matlab notes - all that is needed was developed in the last homework assignment.**

    The nonlinear system $\begin{cases} \dfrac{dx}{dt} = x - y^2 \\ \dfrac{dy}{dt} = 2x^2 - 2y \end{cases}$ has one stable critical point. Find and classify it - by hand, if you want to - and then use the CAS to graph the solution in the xy - phase plane with intial conditon $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix}$. Choose your t-interval so that the picture shows clearly the type of the critical point. What happens if $x(0) = 2$ instead of 1.1? Why does this happen? You might want to investigate the graph when $x(0) = 1.8$ before answering.

Brief answers

1. A) 0, 1, 2 are critical points
   B) x = 0 is unstable (or semistable), x = 1 is stable, and x=2 is unstable.
   C) 1

2. New equation: $\dfrac{dx}{dt} = kx(M - x) + A$ New limiting population $= \dfrac{kM + \sqrt{k^2M^2 + 4Ak}}{2k}$

3. a) 0 is unstable, 10 is stable b) $a < 25$ c) 8 tons

4. $(0.5, 1)$ is a stable spiral.

5. $(1,1)$ is a stable node; $(1,-1)$ and $(2,2)$ are saddle points; $(2,-2)$ is an unstable spiral.

6. $(0,0)$ is an unstable node; $(10,0)$ and $(0,16)$ are saddle points; $(4,12)$ is a stable node.

7. $(1,1)$ is unstable but with unclassifiable type; $(1,0)$ has unclassifiable stability and type.

8. $(0,0,0)$ is unstable because the corresponding eigenvalues are 1, -1, and 1. $(-1,1,1)$ is also unstable since the eigenvalues are -1, and $1 \pm 2i$. (Note: this is a **brief** answer - tell me more precisely why the eigenvalues lead to the stability in these two cases.)

9. $\lambda = n^2$ are the eigenvalues and $x_n = \sin(nt)$ are the corresponding eigenfunctions for $n = 1, 2, 3, \ldots$.

10. $\lambda = n^2\pi^2$ are the eigenvalues and $x_n = \cos(n\pi t)$ are the corresponding eigenfunctions for $n = 0, 1, 2, 3, \ldots$.

11. $\lambda = 0$ has eigenfunction $x_0(t) = t$; the other eigenvalues are all $\lambda > 0$ not equal to $(\pi/2 + n\pi)^2$, $n = 0, 1, 2 \ldots$ with corresponding eigenfunctions $x_\lambda(t) = \dfrac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}\cos(\pi\sqrt{\lambda})}$; and all $\lambda < 0$ with eigenfunctions $x_\lambda(t) = \dfrac{\sinh(\sqrt{-\lambda}t)}{\sqrt{-\lambda}\cosh(\pi\sqrt{-\lambda})}$.

12. $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$

13. $u(x,t) = 3e^{-t}\cos(x) + e^{-9t}\cos(3x)$

14. $u(x,t) = 5e^{-3t}\sin(x) + 2e^{-75t}\sin(5x)$

15. $u(x,t) = 1 + 2e^{-0.1t}\cos(x)$

16. $u(x,t) = Ce^{\lambda(t+x)}$ for any real λ.

17. Many possible answers; for instance, $u(x,t) = Ae^x + Be^t$, or any linear combination of $u_\lambda = e^{\lambda x}e^{(1-\lambda)t}$