20 points Math 220 Homework 9 Chapter 6 18 problems

Some problems are from "Notes on Diffy Q's, Differential Equations for Engineers" by Jiri Lebl.

You may use this table of Laplace Transforms unless asked to use the definition instead. I use $H(t)$ for the heaviside function, not $u(t)$ that is used in Lebl's book. The heaviside function is also known as the unit-step function.

$f(t)$	$F(s)$		$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$		$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$		$t^n f(t)$	$(-1)^n F^{(n)}(s)$
e^{at}; $e^{at}f(t)$	$\frac{1}{s-a}$; $F(s-a)$		$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$f(t-a)H(t-a)$	$e^{-as}F(s)$		$\cos(at)$	$\frac{s}{s^2+a^2}$
t^n	$\frac{n!}{s^{n+1}}$		$\sin(at)$	$\frac{a}{s^2+a^2}$
$f(t)\delta(t-a)$	$f(a)e^{-as}$		$\cosh(at)$; $\sinh(at)$	$\frac{s}{s^2-a^2}$; $\frac{a}{s^2-a^2}$

Use Laplace Transforms to solve the following equations.

1. $q''(t) + 6q'(t) + 8q(t) = 70e^{3(t-1)}H(t-1)$ if $q(0) = 1$ and $q'(0) = 2$.
2. $y''(t) + 9y(t) = \cos(3t)$ if $y(0) = 2$ and $y'(0) = 5$.
3. $x''(t) + 4x(t) = \sin(t)H(t-2\pi)$ if $x(0) = 1$ and $x'(0) = 0$.

4. Find the Laplace Transforms of the following functions.

   a) $k(t) = te^{-t}$ b) $p(t) = (t^2 - 1)H(t-2)$

5. Find the Inverse Laplace Transforms of the following functions.

a) $H(s) = \dfrac{1}{s^2 - 6s + 10}$ b) $P(s) = \dfrac{e^{-\pi s}}{s^2 + 1}$ c) $K(s) = \dfrac{8}{s^3(s+2)}$

6. Write $g(t) = \begin{cases} t & \text{if } 0 \le t < 2 \\ 0 & \text{if } t \ge 2 \end{cases}$ as a linear combination of translates of H(t) and then find its transform.

7. Solve $f(t) + \int_0^t f(\tau)\, d\tau = 1$.

8. Solve $y'(t) = 1 - \sin(t) - \int_0^t y(\tau)\, d\tau$ if $y(0) = 0$.

9. Compute $(f * g)(t)$ if $f(t) = \cos(t)$ and $g(t) = e^{-t}$.

10. Use convolution to solve $x''(t) + x(t) = \sin(t)$ if $x(0) = 0 = x'(0)$.

Solve the following if $x(0) = 0 = x'(0)$. The solution is called the "impulse response".

11. $x''(t) = \delta(t)$.

12. $x'(t) + ax(t) = \delta(t)$

Compute.

13. $\mathscr{L}^{-1}\left(\dfrac{s^2}{s^2+1}\right)$.

14. $\mathscr{L}^{-1}\left(\dfrac{3s^2 e^{-s} + 2}{s^2}\right)$.

15. $\mathscr{L}\left((t^2 - 9)\delta(t-5)\right)$.

16. $\mathscr{L}\left([2H(t-1) + 4H(t-4)]\,\delta(t-5)\right)$.

17. If Θ is a linear operator for a second order differential equation so that $\Theta(x(t)) = \delta(t)$ with $x(0) = 0 = x'(0)$ has the solution $x(t) = \cos(t)$, then what is the solution to $\Theta(x(t)) = \sin(t)$ with $x(0) = 0 = x'(0)$?

18. CAS problem (3 points): use a CAS device for the following. Turn in a pdf file as specified in the syllabus. **Matlab notes for this lab follow the brief answers.**

A) Solve $y'' + 4y = t^2\delta(t-1)$ if $y(0) = 1$ and $y'(0) = 2$ using the **dsolve** command. The real solution is written using e^{it} and does not fit onto a single line (and hence won't completely print onto a pdf.) Use **vpa(simplify(sol),2)** so that the answer will fit on a line.

B) $y'' + 4y = t^2 H(t-5)$ if $y(0) = 0$ and $y'(0) = 0$. Use **simplify(sol)** so the answer will fit on a line.

C) What does the matlab function **sign(t)** mean in this context? Specify the possible outputs for corresponding inputs.

Brief Answers

1. $q(t) = 3e^{-2t} - 2e^{-4t} + H(t-1)(5e^{-4(t-1)} - 7e^{-2(t-1)} + 2e^{3(t-1)})$

2. $y(t) = 2\cos(3t) + \left(\frac{t+10}{6}\right)\sin(3t)$

3. $x(t) = \cos(2t) + \frac{\sin(t)}{3}H(t-2\pi)(1-\cos(t))$

4. a) $\frac{1}{(s+1)^2}$ b) $e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right)$

5. a) $h(t) = e^{3t}\sin(t)$ b) $p(t) = -\sin(t)H(t-\pi)$ c) $k(t) = 2t^2 - 2t + 1 - e^{-2t}$

6. $g(t) = t[H(t) - H(t-2)]$ and $G(s) = \frac{1 - e^{-2s}(2s+1)}{s^2}$

7. $f(t) = e^{-t}$

8. $y(t) = \sin(t)\left(1 - \frac{t}{2}\right)$

9. $(f * g)(t) = 0.5(\cos(t) + \sin(t) - e^{-t})$

10. $x(t) = \frac{1}{2}(\sin(t) - t\cos(t))$

11. $x(t) = t$

12. $x(t) = e^{-at}$

13. $\delta(t) - \sin(t)$

14. $3\delta(t-1) + 2t$

15. $16e^{-5s}$

16. $6e^{-5s}$

17. $x(t) = 0.5t\sin(t)$

MatLab notes are on the next page.

Notes for CAS HW#9

Use the dsolve command as you have in the past, but you will need two functions, dirac, and heaviside.

```
syms y(t)
eqnt=diff(y,t,2)+4*diff(y,t)+3*y==cos(t)*dirac(t-2);
yp(t)=diff(y,t);
conds=[y(0)==0,yp(0)==1];
sol=dsolve(eqnt,conds);
vpa(simplify(sol),2)
```

ans = $41.0\,e^{-3.0\,t} - 0.27\,e^{-1.0\,t} - 0.77\,e^{-1.0\,t}\,\mathrm{sign}(t-2.0) + 42.0\,e^{-3.0\,t}\,\mathrm{sign}(t-2.0)$

Here is similar work using a heaviside function.

```
clear
syms y(t)
eqnt=diff(y,t,2)+4*diff(y,t)+3*y==t^2*heaviside(t-2);
yp(t)=diff(y,t);
conds=[y(0)==0,yp(0)==1];
sol=dsolve(eqnt,conds);
vpa(simplify(sol),2)
```

ans = $0.5\,e^{-1.0\,t} - 0.5\,e^{-3.0\,t} + 0.25\,\left(\mathrm{sign}(t-2.0)+1.0\right)\,\left(t^2-2.0\,t+2.0\right) - 9.3\,10^{-3}\,\left(\mathrm{sign}(t-2.0)+1.0\right)\,(9.0\,t^2$