# Introduction

A differential equation (DifEq) has derivatives of an unknown function.

For example: $q''(t) + 3q'(t) + 2q(t) = e^{-t}$. We want to find $q(t)$. The order of this DifEq is 2. What is the definition of the order of a DifEq?

Guessing and Checking, or "solving by inspection," is our first method for solving DifEq's. Find all the solutions to $\frac{dy}{dx} = 2y$ by inspection.

An "operator" is a function with domain consisting of functions; $D = \frac{d}{dx}$ is a "linear operator," a linear tranformation that is an operator. We can write $\frac{dy}{dx} = 2y$ using operator notation as $D(y) = 2y$. So 2 is an eigenvalue of $D$. What is a basis for the corresponding eigenspace?

Solve $\dfrac{d^2x}{dt^2} = -9x$ by inspection.

Rewrite the equation using a linear operator. What is a basis for the solution set?

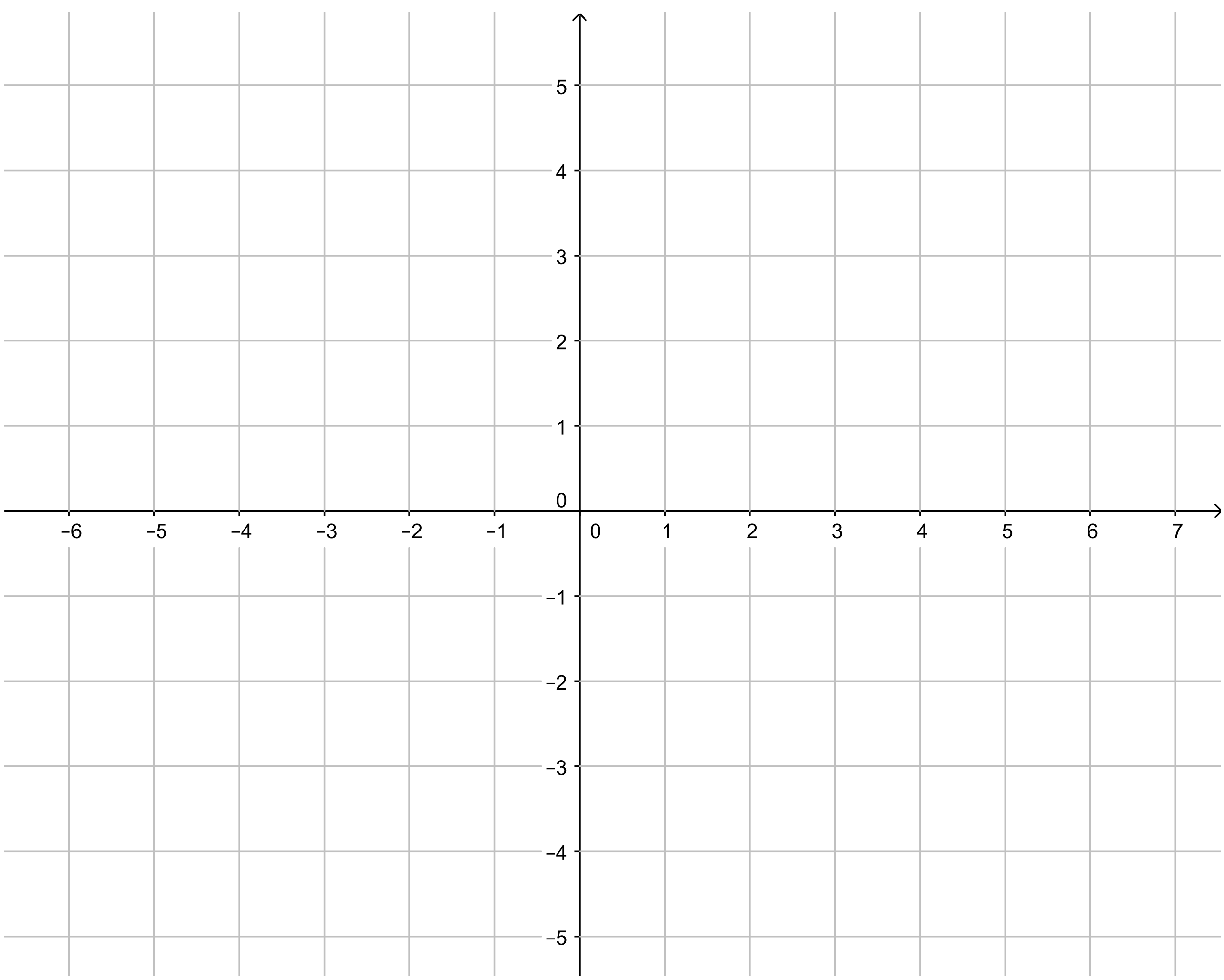
We can rewrite $\dfrac{d^2x}{dt^2} = -9x$ as a system of first order equations if we let $x_1 = x$ and $x_2 = x'$. How do I do it?

Any DifEq that can be written in the form $\vec{X}'(t) = A(t)\vec{X}(t) + \vec{f}(t)$ where A is an nxn matrix is a linear DifEq of order n. Our example produced an $A$ with constant coefficients.

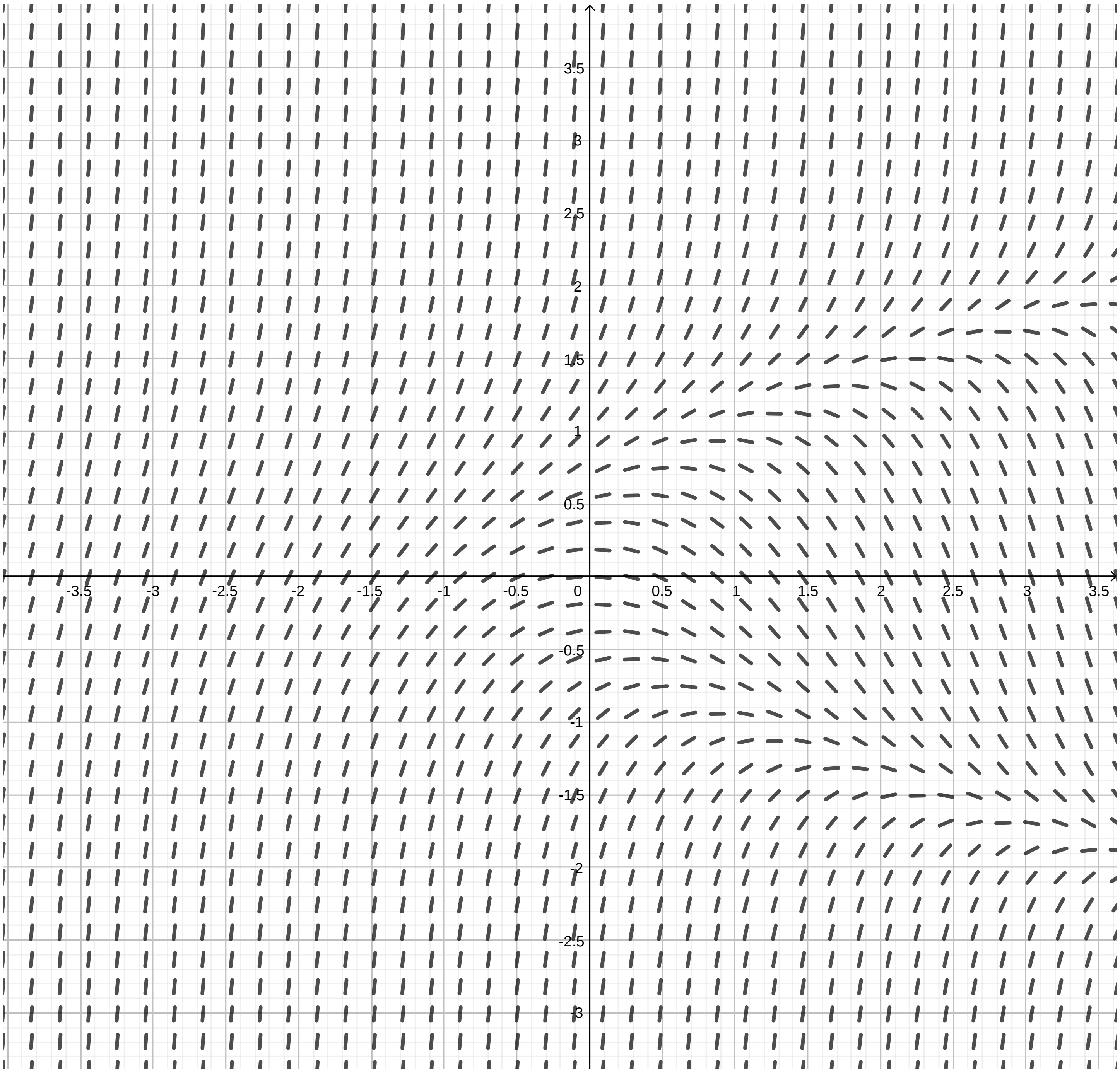
# Slope Fields

A first order ODE is of the form $y' = f(x, y)$. Usually we can only estimate solutions using numerical methods such as **ode45** in Matlab. A **slope field** is the start to all of these methods.

Example: $y' = y^2 - x$, $y(0) = 0$ is an "initial value problem" (IVP). Draw the slope field for $y' = y^2 - x$ and estimate the graph of the solution to the IVP. Draw several "isoclines" to help expedite your work.



Of course, no one draws slope fields by hand anymore unless they are students or teachers. The online graphing calculator "Geogebra" produces this slope field for $y' = y^2 - x$. Draw the solution to $y' = y^2 - x$, $y(0) = 0$ on this graph.




Is this the only solution when $y(0) = 0$? How do we know the solution even exists? These questions are answered by the Existence-Uniqueness Theorem.

# Existence-Uniqueness Theorem[1]

**If $y'(x) = f(x, y)$, $y(a) = b$ is an IVP so that**

**1)** $f(x, y)$ **is continuous in a box about** $(a, b)$**, and**
**2)** $f_y(x, y)$ **is continuous in a box about** $(a, b)$

**then there exists a unique solution for the IVP in some box about** $(a, b)$**.**

Example: Show $y' = y^2 - x$, $y(0) = 0$ has a unique solution.

Example: Can we find more than one solution for any of the following equations?

1) $y' = 4x^{1/3}$, $y(0) = 0$

2) $2y' = 3y^{1/3}$, $y(0) = 0$.

[1]The proof constructs a sequence of functions, Picard interations, that converges to a solution. Convergence depends on the "completeness" of the space of functions; uniqueness is easier to prove.

# Separation of Variables

If $\frac{dy}{dx} = f(x)g(y)$, then $\int \frac{dy}{g(y)} = \int f(x)\,dx$.

How do I prove this statement?

Example: Solve $y' = \frac{x\sin(x)}{\cos^2(y)}$, $y(\pi) = 0$ **implicitly**.

The Malthusian model [2] says the rate of change of population is proportional to the population, $P$.

$$\frac{dP}{dt} = kP$$

This model works well if there are no constraints, for instance, when mold grows on a slice of bread. When constraints are present, as when the mold devours most of the bread, a new model is needed. The **logistic model**, where the constant of proportionality becomes a linear function of $P$, often works well in this case.

Example: Solve $P' = (9 - 2P)P$ **explicitly**. We will need to review how to perform partial fractions fast using the "cover-up" method. [3]

[2] Thomas Robert Malthus 1766-1834

[3] I found this method from a MIT free online lecture.