First Order Linear Equations

The standard form is

$$\frac{dy}{dx} + p(x)y = f(x) \qquad (*)$$

One way to solve this equation is to find an integrating factor $u(x)$ that, when multiplied on both sides of the above equation, completes the product rule on the left hand side of the equation:

$$\frac{d(uy)}{dx} = u\frac{dy}{dx} + up(x)y$$

How do we find $u(x)$?

Now multiply both sides of $(*)$ by u and solve for y.

Example: Suppose a tank with capacity 500 liters starts with 200 liters of brine that has 20 grams of salt in it. Brine containing five grams of salt per liter enters the tank at three liters per minute, instantly mixes into a homogeneous brine within the tank, and exits the tank at two liters per minute. How much salt is in the tank at any time t? What is the range for t?

Example: A circuit with a 100 Ω resistor, a 0.05 F capacitor, and an impressed voltage $E(t) = 200\cos(t)$ V has an initial charge on the capacitor of 10 C. Find the charge $q(t)$ on the capacitor at any time t.

Kirchoff's Laws give the equation to be[1]

$$Rq'(t) + \frac{1}{C}q(t) = E(t)$$

I will use this example as an opportunity to use complex integration of[2]

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

.

[1] or $Ri'(t) + \frac{1}{C}i(t) = E'(t)$ if $i(t)$ is the current in the circuit at any time t. $i(t) = q'(t)$.

[2] Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies e^{i\theta} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos(\theta) + i\sin(\theta)$.

Substitution

Example: Bread is removed from an oven at 250°C and put on a rack to cool in a room with 25°C constant temperature. After one minute the bread is 150°C. Find the temperature at any time t.

We use Newton's Law of Cooling (or heating):

$$\frac{dT}{dt} = k(T - T_s) \qquad (*)$$

where T_s=constant ("static") temperature of the surroundings and k is a constant. Notice $k < 0$. Why?

Substituting $u = T - T_s$ means $\frac{du}{dt} = \frac{d(T - T_s)}{dt} = \frac{dT}{dt}$ so that $(*)$ becomes

$$\frac{du}{dt} = ku$$

an equation that we can solve by inspection.

The rest of the problem is from precalculus. Solve it.

Bernoulli's Equation[3]

The form is almost linear:

$$y' + p(x)y = f(x)y^n$$

Dividing by y^n we see that a substitution of $u = y^{1-n}$ should make a linear equation. What do we need to calculate to complete the substitution?

Find an explicit solution for $xy' + y = x^4 y^4$; $y(1) = 0.5$.

[3]For a frictional force application see https://math.stackexchange.com/questions/344839/how-bernoulli-differential-equation-arise-naturally.

Homogeneous Substitution

The form is :

$$y' = F\left(\frac{y}{x}\right)$$

Use the substitution of $u = \dfrac{y}{x}$ to make a separable equation[4]. What do we need to calculate to complete the substitution?

Find an implicit general solution for $y' = \dfrac{y^2 + 3xy + x^2}{x^2}$.

[4]More generally, $f(x, y)$ is a homogeneous function of degree n if $f(ax, by) = a^n f(x, y)$; if $A(x, y)$ and $B(x, y)$ are homogeneous functions of the same degree, then $A(x, y)dx + B(x, y)dy = 0$ becomes separable if we substitute $u = \dfrac{y}{x}$.

Exact Equations (not in our book)

Multivariate Calculus defined $\vec{F}(x,y) =< M(x,y), N(x,y) >$ to be a gradient field if it equaled the gradient of a potential function, $\vec{F}(x,y) = \nabla\Phi$. Then $\Phi_x = M$ and $\Phi_y = N$; if M and N have continuous derivatives, then the mixed partials of Φ are equal so that we get the curl test:

$$M_y = N_x$$

when $\vec{F}$ is a gradient field.

But we can also use the chain rule and get

$$\frac{d\Phi(x,y(x))}{dx} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\cdot\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx}$$

Then the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad (*)$$

$$\Longrightarrow \frac{d\Phi(x,y(x))}{dx} = 0$$

has the implicit general solution $\Phi(x,y) = C$.

Typically $(*)$ is written using differential notation so that the left side looks like the integrand of a line integral. Therefore we define an **exact equation** to be

$$M(x,y)dx + N(x,y)dy = 0$$

if

$$M_y = N_x$$

and has implicit general solution

$$\Phi(x,y) = C$$

where Φ is a potential function for $\vec{F} =< M, N >$.

You learned at least two of the following methods for finding the potential function: 1) by inspection; 2) integrating-differentiating; 3) using a line integral. Please review and learn again if necessary: if you lost your old notes and book, you can visit my old math 200 course and look at those notes.

Example: Find a constant c so that $(2 + cx^2y^2)dx + 2x^3ydy = 0$, $y(1) = 2$ is an exact equation. Then find an explicit solution.