Nonlinear First Order Autonomous Equations

Equations of the form $\frac{dy}{dx} = f(y)$ are called **autonomous**. If $f(y)$ is nonlinear we usually find a numerical estimate for y. Unfortunately many such systems are sensitive to initial conditions; that means a small change in the initial condition can result in a large change to long-term behavior. We seek answers that are reasonable in the long-term. The following method does that, and can be generalized to systems of equations.

Two observations about $\frac{dy}{dx} = f(y)$:

1) If k is constant and $f(k) = 0$, then $y = k$ is a solution. k is then called a **critical point**, and $y = k$ is called an **equilibrium solution**.

2) The slope field does not change on any horizontal line; thus we may use the one-dimensional y-axis to summarize the long term behavior of the slope field. This is called a **phase diagram** for the equation.

Example: Find the critical points of $y' = 7(y-2)(y+6)$, draw the phase line, and classify each critical point as stable or unstable. Sketch a complete set of typical solution curves in a slope field that includes the equilibrium solutions.

Example: Find the critical points of $y' = |y|$, draw the phase line, and classify each critical point as stable or unstable. Sketch a complete set of typical solution curves in a slope field that includes the equilibrium solutions.

Example: A salmon farmer wants to maximize a sustained harvest rate of α tons of salmon per week. The number of tons of fish on the farm in the absence of harvesting obeys $\dfrac{dP}{dt} = -P^2 + 6P$ where t is the number of weeks in elapsed time.

a) Write the equation if the harvest rate α is included. What rates will allow for an approximate constant mass of salmon on the farm?

Tactics: Find and classify equilibrium points. If there are stable critical points, then constant mass is possible.

b) If the harvest rate is $\alpha = 5$ tons per week over a long period of time, what is the approximate constant mass of fish on the farm? Defend your answer.