Linear Homogeneous DiffEq with Constant Coefficients

n^{th} order:

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dx^i} = 0 \qquad (\frac{d^0 y}{dx^0} = y)$$

We can always create a linear operator L to express these equations as $L(y) = 0$, so the solution set is a null space.

For instance[1], if $L = aD^2 + bD + c$, then $ay'' + by' + cy = 0$ can be written as $L(y) = 0$.

Hence we need only find a basis for the null space to find the general solution. Any such basis is called a **fundamental set**.

We may use a generalized Existence-Uniqueness theorem to prove the solution sets of an nth order difeq are n-dimensional vector spaces, so

to find the general solution we must find n linearly independent solutions.

We solve $ay'' + by' + cy = 0$ by guessing: what should we guess? The correct guess will produce the **characteristic equation**.

There are three cases: two distinct real roots, two complex roots, and one real repeated root.

[1]Recall $D = \frac{d}{dx}$

Example: Solve $y'' + 4y' + 3y = 0$; $y(0) = 1$, $y'(0) = 2$.

Example: Solve $y'' + 2y' + 5y = 0$. This means find all of the **real**, not complex, solutions.

Here is some review of the complex arithmetic just used:

If $z = a + bi$ then the **conjugate** is $\bar{z} = a - bi$. $a = Re(z)$ is the **real part** of z , and $b = Im(z)$ is the **imaginary part**. b is a real number notwithstanding its name. Notice that

$$Z = e^{(a+bi)x} = e^{ax+bxi} = e^{ax}\cos(bx) + ie^{ax}\sin(bx)$$

so that

$$Re(Z) =$$

and

$$Im(Z) =$$

Also,

$$\overline{Z} = \overline{e^{(a+bi)x}} = e^{ax}\left(\cos(bx) - i\sin(bx)\right) = e^{(a-bi)x} = e^{\overline{(a+bi)}x}$$

so Z and $\overline{Z}$ are both solutions to the equation with characteristic roots $a \pm b$.

Then

$$\frac{Z + \overline{Z}}{2} = Re(Z)$$

and

$$\frac{Z - \overline{Z}}{2i} = Im(Z)$$

.

So $Re(Z)$ and $Im(Z)$ are linear combinations of two complex solutions and hence solutions as well. They are also independent since one is not a multiple of the other. Therefore they form a basis for the real solution set[2].

If the characteristic roots are $a \pm bi$ then

$$y = e^{ax}\left[c_1\cos(bx) + c_2\sin(bx)\right]$$

[2] Z and $\bar{Z}$ are independent if $Im(Z) \neq 0$. If we identify Z and $\bar{Z}$ with vectors in $\mathbb{C}^2$, then they form a basis B, and the B-coordinates of the two solutions are $\left(\frac{1}{2}, \frac{1}{2}\right)_B$ and $\left(\frac{1}{2i}, \frac{-1}{2i}\right)_B$. $\begin{vmatrix} 1/2 & 1/2 \\ 1/2i & -1/2i \end{vmatrix} \neq 0$, so the vectors are independent.

Example: Solve $y'' + 4y' + 4y = 0$.

Hint: Use operation notation to show guess is a solution.

Higher order equations are handled the same way; if a root is repeated more than once, an extra factor of x is added each time.

Solve

$$y^{(5)} + y^{(3)} = 0$$

Write the general solution for a seventh order homogeneous linear equation with constant coeffcients if the characteristic equation has roots $r = 2, 2, 2, 3 \pm 4i, 3 \pm 4i$.