Independent Functions

If $f(x)$ and $g(x)$ are independent functions, then the only zero combination of the two is the trivial one:

$$c_1 f(x) + c_2 g(x) = 0 \implies c_1 = 0 = c_2$$

Differentiation gives us a system of equations:

$c_1 f(x) + c_2 g(x) = 0$ differentiated on both sides gives $c_1 f'(x) + c_2 g'(x) = 0$, so

$$\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then we can say $f(x)$ and $g(x)$ are independent iff $\exists x_0 \in \mathbb{R}$ so that

$$W(x_0) = \begin{vmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{vmatrix} \neq 0$$

W is called the **Wronskian**. [1]

If $f(x)$ and $g(x)$ also happen to be solutions to the same differential equation that meets the hypotheses of the existence-uniqueness theorem, then either $f(x) = cg(x)$ for all x and for some c, or $f(x) \neq cg(x)$ for all x and all c. Therefore, in this case:

$$W(x_0) \neq 0 \implies W(x) \neq 0 \;\; \forall \, x$$

If $f(x)$ and $g(x)$ are a basis for a solution set of a differential equation, then

$$\Phi(x) = \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

is a **fundamental matrix** of the differential equation and

$$W = \left|\Phi\right|.$$

[1] Hoene Wronski 1778-1853.

Example: Show $\cos(x)$, x, and $x\cos(x)$ are independent functions but are not simultaneously solutions of a differential equation that satisfies the existence-uniqueness theorem.

Example: Decide if e^x, e^{-x}, and $\cosh(x)$ are independent functions. Defend your answer.

There are always different bases for the same vector space. So, for instance, the solution set for $y'' = y$ has a basis e^x, e^{-x}. Another important one is $\cosh(x)$, $\sinh(x)$.