Independent Functions

If $f(x)$ and $g(x)$ are independent functions, then the only zero combination of the two is the trivial one:

$$c_1 f(x) + c_2 g(x) = 0 \implies c_1 = 0 = c_2$$

Differentiation gives us a system of equations:

$c_1 f(x) + c_2 g(x) = 0$ differentiated on both sides gives $c_1 f'(x) + c_2 g'(x) = 0$, so

$$\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then we can say $f(x)$ and $g(x)$ are independent iff $\exists x_0 \in \mathbb{R}$ so that

$$W(x_0) = \begin{vmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{vmatrix} \neq 0$$

W is called the **Wronskian**. [1]

If $f(x)$ and $g(x)$ also happen to be solutions to the same differential equation that meets the hypotheses of the existence-uniqueness theorem, then either $f(x) = cg(x)$ for all x and for some c, or $f(x) \neq cg(x)$ for all x and all c. Therefore, in this case:

No

$$W(x_0) \neq 0 \implies W(x) \neq 0 \;\; \forall x$$

If $f(x)$ and $g(x)$ are a basis for a solution set of a differential equation, then

$$\Phi(x) = \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

is a **fundamental matrix** of the differential equation and

of order 2.

$$W = |\Phi| .$$

[1]Hoene Wronski 1778-1853.

Example: Show $\cos(x)$, x, and $x\cos(x)$ are independent functions but are not simultaneously solutions of a differential equation that satisfies the existence-uniqueness theorem.

$$W(x) = \begin{vmatrix} \cos(x) & x & x\cos(x) \\ -\sin(x) & 1 & \cos(x) - x\sin(x) \\ -\cos(x) & 0 & -2\sin(x) - x\cos(x) \end{vmatrix}$$

$$W(0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = 0$$

$$W(\pi) = \begin{vmatrix} -1 & \pi & -\pi \\ 0 & 1 & -1 \\ 1 & 0 & \pi \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ 0 & \pi \end{vmatrix} - 0 + (1) \begin{vmatrix} \pi & -\pi \\ 1 & -1 \end{vmatrix}$$

$$= -\pi + (-\pi + \pi) = -\pi \neq 0.$$

$\therefore$ $W(\pi) \neq 0 \to$ the functions are independent, and then

$W(0) = 0 \to$ the functions cannot satisfy the same 3^{rd} order diff eq.

Example: Decide if e^x, e^{-x}, and $\cosh(x)$ are independent functions. Defend your answer.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

$\Rightarrow \cosh(x) \in \text{span } e^x, e^{-x}$

$\Rightarrow$ the set of functions is dependent.

Note: $W(x) = \begin{vmatrix} e^x & e^{-x} & \cosh(x) \\ e^x & -e^{-x} & \sinh(x) \\ e^x & e^{-x} & \cosh(x) \end{vmatrix}$

$$W(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 0 + 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 + (-2) = 0$$

There are always different bases for the same vector space. So, for instance, the solution set for $y'' = y$ has a basis e^x, e^{-x}. Another important one is $\cosh(x)$, $\sinh(x)$.

$y'' = y \Rightarrow y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r^2 = 1 \Rightarrow r = \pm 1$

$y = C_1 e^x + C_2 e^{-x} = A_1 \cosh(x) + A_2 \sinh(x)$

i.e. $\text{span } e^x, e^{-x} = \text{span } \cosh(x), \sinh(x)$.