Unforced Mechanical Oscillations

A spring, mass, and dashpot on a frictionless track is a great ideal mechanical model for oscillations. I like to use a horizontal track so that I can ignore standard gravity. Draw two pictures, one in which the mass is in equilibrium and another where the mass is not in equilibrium. Let $x(t)$ = displacement from equilibrium at time t in seconds.

$x(t)$ depends on three physical laws:

1) **Hooke's Law**: $F_s = -kx$; The force a spring exerts on a mass x units from equilibrium is proportional to and opposite of the displacement.

2) **Drag**: $F_D = -\alpha x'$; The damping force on a moving mass is proportional to the velocity opposite motion [1].

3) **Newton's second law** gives total force $= mx'' = F_s + F_D$ which becomes

$$mx'' + \alpha x' + kx = 0$$

We have three cases: **over damped** if the two roots are real and distinct; **critically damped** if the two roots are equal; **under damped** if the two roots are complex.

[1] The exponent on x' may be something other than one depending on the situation; we will assume an exponent of one.

A one kilogram mass is attached to a dashpot with damping coefficient 3 and to a spring that requires a 4 Newton force to hold it 2 meters from equilibrium. Find the displacement from equilibrium at time t in seconds if the mass is initially stretched 2 meters and kicked so that it has an initial velocity of 1 meter/sec. Sketch the graph of the solution. Is it over, under, or critically damped?

A one kilogram mass is attached to a dashpot with damping coefficient 4 and to a spring that requires a 8 Newton force to hold it 2 meters from equilibrium. Find the displacement from equilibrium at time t in seconds if the mass is initially compressed 2 meters and kicked so that it has an initial velocity of 1 meter/sec. Sketch the graph of the solution. Is it over, under, or critically damped?

A one kilogram mass is attached to a dashpot with damping coefficient 4 and to a spring that requires a 26 Newton force to hold it 2 meters from equilibrium. Find the displacement from equilibrium at time t in seconds if the mass is initially stretched 1 meter from equilibrium and kicked so that it has an initial velocity of 4 meter/sec. Sketch the graph of the solution. Is it over, under, or critically damped? Find the phase shift, δ, angular frequency, ω, and amplitude, A, of the solutions[2].

[2] $c_1 \cos(\omega\theta) + c_2 \sin(\omega\theta) = < c_1, c_2 > \cdot < \cos(\omega\theta), \sin(\omega\theta) > = \sqrt{c_1^2 + c_2^2} \cos(\omega\theta - \delta)$ where δ is the angle between the x-axis and $< c_1, c_2 >$.

The primary use of the spring-mass-dashpot system is to give a mechanical analogue for the LRC circuit. Kirchoff's Voltage Law says the impressed voltage on a circuit equals the voltage drops across each device of the circuit. We have already discussed a circuit with a resistor and a capacitor. If we add an inductor, we have an extra voltage drop of Lq'' where $L =$ a measure of inductance in Henrys.

The new equation is

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

L is similar to mass, R to drag, and $\frac{1}{C}$ to spring constant. (picture)

Example: Find the charge on the capacitor at time t for the LRC circuit with $L = 0.05$ h, $R = 2\,\Omega$, $C = 0.01$ F if $q(0) = 0.1$ C and $i(0) = 0$ A. (Notice, no impressed voltage is given.)