Nonhomogeneous Linear ODE's

Solve $L(y) = 0$ if $L = D^2 + 4D + 3$. This is a **homogeneous** equation because of the 0 on the right side.

If $L(y) = f(x) \neq 0$, then the equation is called **nonhomogeneous**. Linear Algebra taught us that the nonhomogeneous system $A\vec{x} = \vec{b}$ has general solution equal to $N(A) + \vec{x}_p$, where $N(A)$ is the null space of A and $\vec{x}_p$ is a particular solution.

In the same way, using the same proof[1], $L(y) = f(x)$ has a general solution of the form $y_c + y_p$ where y_c is the general solution for $L(y) = 0$, called the complimentary or characteristic or homogeneous solution and y_p is a particular solution.

Therefore to solve a nonhomogeneous linear equation we must find the general homogeneous solution and one particular solution to the nonhomogeneous equation. We will discuss two different ways to find a particular solution: 1) guess and check (also known as undetermined coefficients), and 2) variation of parameters.

[1] $L(y_c + y_p) = L(y_c) + L(y_p) = 0 + f(x) = f(x) \implies y_c + y_p$ is in the solution set; and, if y is any solution, then $L(y - y_p) = L(y) - L(y_p) = f(x) - f(x) = 0 \implies y - y_p \in y_c \implies y \in y_c + y_p$ so that $y_c + y_p$ contains all of the solutions.

Example: $L(y) = x^2$, where $L = D^2 + 4D + 3$ as above. We will use **undetermined coefficients** which asks "what linear combination of the derivatives of x^2 could be a solution?

Undetermined coefficients only works for $L(y) = f(x)$ if $f(x)$ has a finite number of nonzero derivatives.

Example: $L(y) = 6$, where $L = D^2 + 4D + 3$ as above.

Example: $L(y) = e^{-x}\sin(2x)$, where $L = D^2 + 4D + 3$ as above. I think complexification is easiest. You may decide guessing $y_p = e^{-x}(A\sin(2x) + B\cos(2x))$ is easiest.

Example: $L(y) = e^{-x}$, where $L = D^2 + 4D + 3$ as above. $y_p = Ae^{-x}$ will not work. Why not? We could guess $y_p = Axe^{-x}$, but let me use this example to introduce variation of parameters. Not only will it work in this case where $f(x)$ overlaps the homogeneous solution, but it also works when $f(x)$ has an infinite number of nonzero derivatives. I will use the method without proof, and then prove the method afterwords.

Matrix algebra allows an easy, but subtle, proof. Recall that we can write $y'' + 4y' + 3y = e^{-x}$ using matrices as

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-x} \end{bmatrix}$$

or

$$\vec{u}\,' = A\vec{u} + \vec{f}$$

$y_c = c_1 e^{-3x} + c_2 e^{-x}$ is the homogeneous solution and so $y_c' = -3c_1 e^{-3x} - c_2 e^{-x}$. This can be expressed using matrices as

$$\vec{u}_c = \begin{bmatrix} y_c \\ y_c' \end{bmatrix} = \begin{bmatrix} e^{-3x} & e^{-x} \\ -3e^{-3x} & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi \vec{c}$$

Conveniently, $\vec{u}_c$ is the set of all homogeneous solutions to $\vec{u}\,' = A\vec{u}$. Verify this.

Hence

$$\Phi' \vec{c} = A \Phi \vec{c} \qquad (*)$$

for any $\vec{c}$.

We then guess that $\vec{u}_p = \Phi \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \Phi \vec{v}$ is a solution where $\vec{v} = \vec{v}(x)$ **varies**, and solve for $\vec{v}$. Substitution gives

$$\vec{u_p}\,' = A\vec{u_p} + \vec{f}$$

$$\implies \Phi' \vec{v} + \Phi \vec{v}\,' = A \Phi \vec{v} + \vec{f}$$

$(*)$ implies $\Phi' \vec{v} = A \Phi \vec{v}$ so that

$$\Phi \vec{v}\,' = \vec{f}$$

You can solve this using any technique you learned in Linear Algebra: row reduction, multiplying by an inverse, or Cramer's rule. Cramer's rule is easy for the 2 x 2 case and gives us

$$\vec{v_1}\,' = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^{-x} & -e^{-x} \end{vmatrix}}{|\Phi|}$$

and

$$\vec{v_2}\,' = \frac{\begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & e^{-x} \end{vmatrix}}{|\Phi|}.$$

In general, we can find a particular solution for $L(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ if y_1 and y_2 form a basis for the homogeneous solutions, and

$$\vec{v_1}\,' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

and

$$\vec{v_2}\,' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Example: Solve $4y'' + 4y = \sec(x)$