Forced Oscillations and Resonance

Ideal Resonance - no damping

If $mx'' + kx = F_0 \cos(\omega t)$, F_0 constant, show that $x_c = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t - \delta)$ if $\omega_0 = \sqrt{\dfrac{k}{m}}$. Recall that ω_0 is the **natural frequency**.

Now complexify and find $x_p = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$.

Consequently, $x(t) = \sqrt{c_1^2 + c_2^2}\cos(\omega_0 t - \delta) + \dfrac{1}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$. As $\omega \to \omega_0$, the particular solution's amplitude approaches infinity resulting in infinite displacement from a small forcing function. This phenomena is called **ideal resonance**, and though damping is always present, many applications of **practical resonance** are found in literature[1]. Two classical applications of resonance: 1) Soldiers do not march over bridges for fear of producing a resonant forcing function that can collapse the bridge; 2) Old radio receivers varied capacitance until the desired forcing function became resonant causing that signal to be amplified while the others were minimized.

Practical Resonance

When damping is present, say $mx'' + \alpha x' + kx = F_0 \cos(\omega t)$, then the homogeneous solution has a factor equal to $e^{-\frac{\alpha}{2m}t}$ and so approaches zero quickly - it is a **transient** solution. Consequently,the particular solution, or **steady state**, determines long-term behavior. Use $2p = \dfrac{\alpha}{m}$ and $\omega_0^2 = \dfrac{k}{m}$ while complexifying to find $x_p = \dfrac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4p^2\omega^2}}\cos(\omega t - \delta)$.

[1]Type in "resonance" at arXiv.org to find 1000 Physics research articles using it.

To find the desired ω that maximizes amplitude, we must minimize $R(\omega) = (\omega_0^2 - \omega^2)^2 + 4p^2\omega^2$. Show the minimum $R(\omega)$ is when $\omega = \sqrt{\omega_0^2 - 2p^2}$ using first semester calculus methods.

Write the formula for resonance for the circuit modeled by

$$Lq'' + Rq' + \frac{1}{C}q = F_0 \cos(\omega t)$$