Linear Systems

Our goal is to solve linear first order systems with constant coefficients. Two unknown functions produce a 2 x 2 system:

$$\begin{cases} x'(t) = ax(t) + by(t) + f_1(t) \\ y'(t) = cx(t) + dy(t) + f_2(t) \end{cases} \implies \vec{u}\,'(t) = A\vec{u} + \vec{f}(t)$$

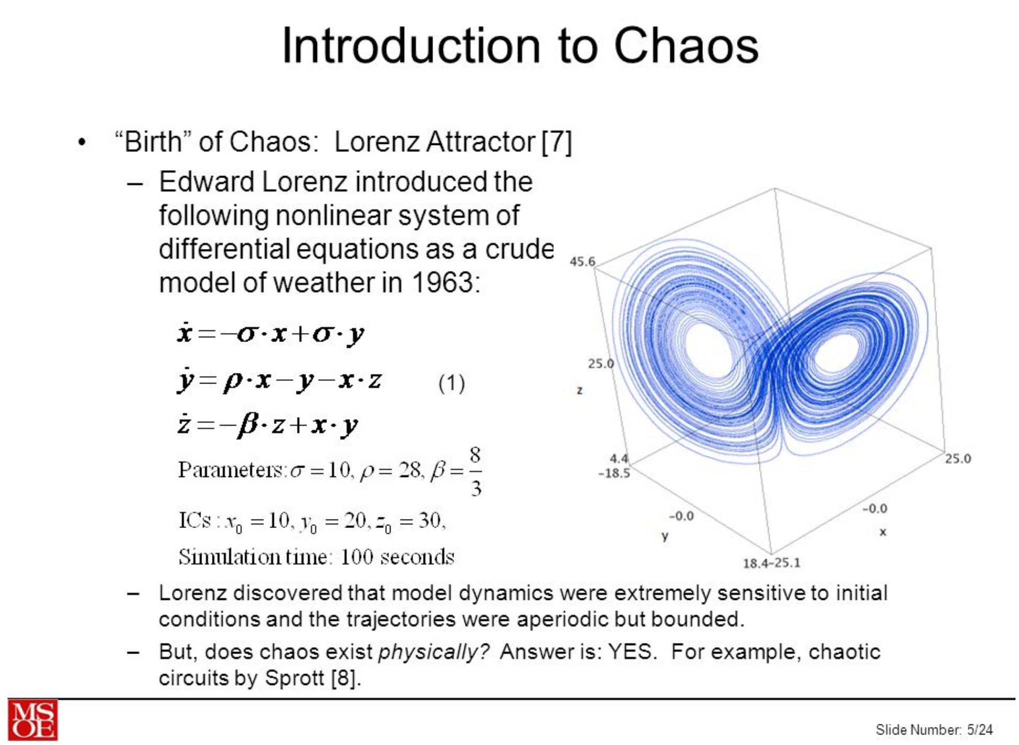
where $\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$. If $\vec{f}(t) = \vec{0}$, the system $\vec{u}\,' = A\vec{u}$ is **homogeneous**; otherwise, it is **nonhomogeneous**.

The most useful systems of equations use many more than two functions. In general we would like to solve n x n systems:

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + f_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + f_2(t) \\ \qquad \ddots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + f_n(t) \end{cases} \implies \vec{u}\,'(t) = A\vec{u} + \vec{f}(t)$$

where $\vec{u}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$.

Linear systems can be used to make conclusions about nonlinear systems such as the Lorenz equations:



Review of Eigenvalues and Eigenvectors

Find the eigenvalues and eigenspaces of the following matrices.

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix}$$

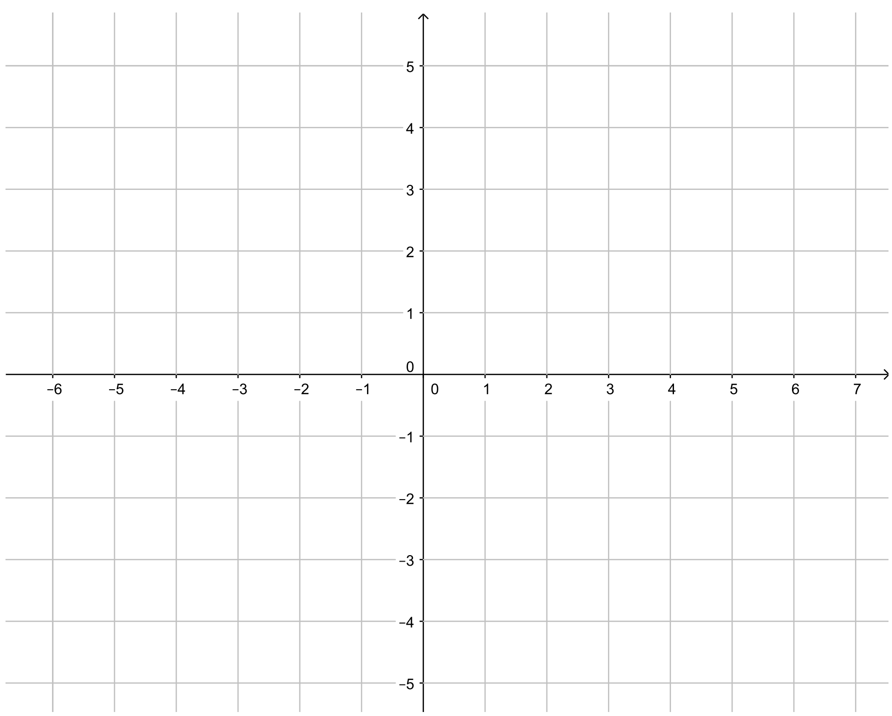
If we have a system of the form $\vec{u}'(t) = A\vec{u}$ then we guess that

$$\vec{u}(t) = e^{\lambda t}\vec{v}$$

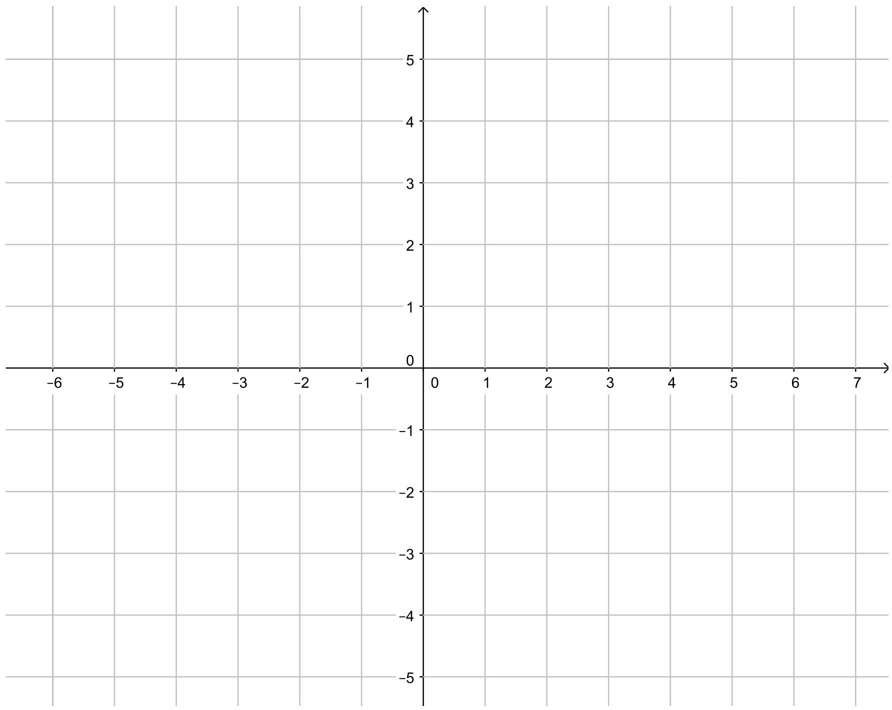
is a solution where $\vec{v}$ is constant. Then how are A, λ, and $\vec{v}$ related?

If A is n x n then the Existence-Uniqueness Theorem will imply the vector space of solutions will be n-dimensional. Consequently, we need only find n independent solutions. We first try to find n linearly independent eigenvectors and form a basis from their corresponding solutions.

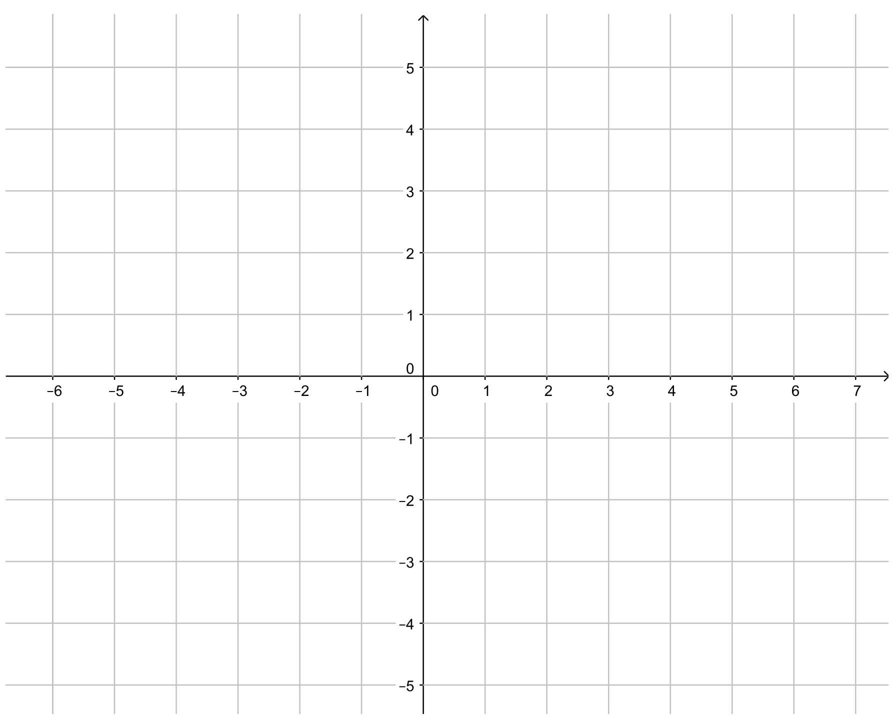
Example Solve $\begin{array}{rcl} x' &=& -2x+2y \\ y' &=& x-3y \end{array}$. Graph the solution in the **phase plane**, the xy-plane in this case. (0,0) is the only critical point of this autonomous system; why? Classify it.



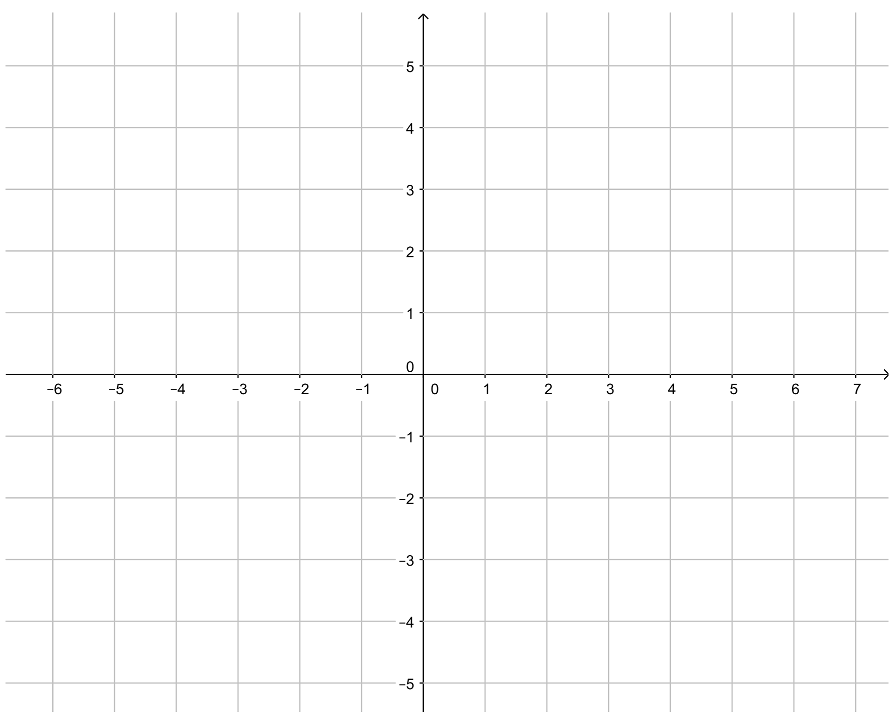
Suppose our system had the same eigenvectors, but the eigenvalues were $\lambda = 4, 1$. Write the solution and its phase diagram. Classify (0,0).



Suppose our system had the same eigenvectors, but the eigenvalues were $\lambda = -4, 1$. Write the solution and its phase diagram. Classify (0,0).



Example Solve $\begin{array}{rcl} x' & = & 2x - y \\ y' & = & x + 2y \end{array}$. Graph the solution in the **phase plane**, the xy-plane in this case. Classify the critical point.



Reviewing: We used $\lambda = 2 + i$ to find our eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 + 0i \\ 0 - i \end{bmatrix}$. Then

$$e^{(2+i)t}\vec{v} = e^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1 + 0i \\ 0 - i \end{bmatrix}$$

so

$$Re\left(e^{(2+i)t}\vec{v}\right) = e^{2t}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

and

$$Im\left(e^{(2+i)t}\vec{v}\right) = e^{2t}\begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} = e^{2t}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

.

Write the solution if $\lambda = m + pi$ and $\vec{v} = \begin{bmatrix} a + bi \\ c + di \end{bmatrix}$.

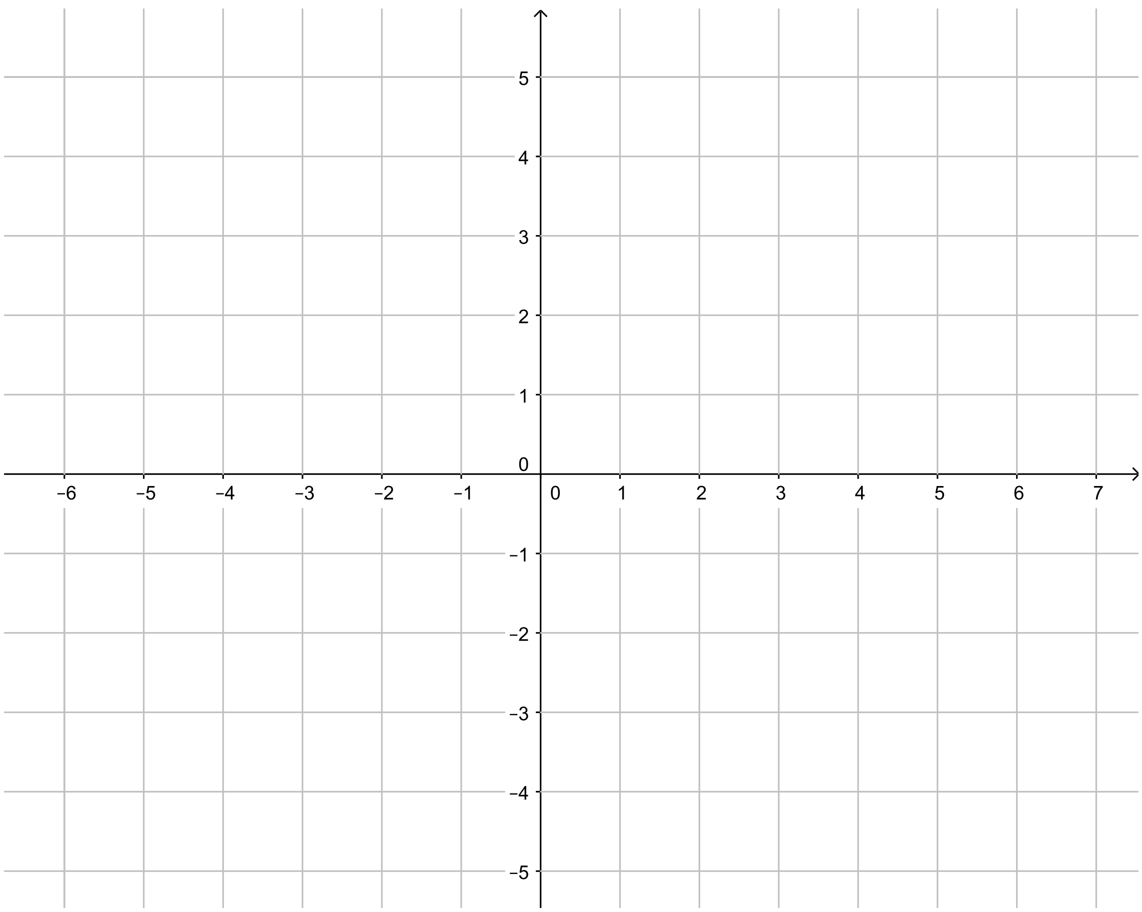
Choose m, p, a, b, c, d so that $(0, 0)$ is a stable spiral turning clockwise.

What if we have a repeated eigenvalue with **algebraic multiplicity** n? The method does not change if the dimension of the eigenspace, the **geometric multiplicity**, is also n.

We say the eigenvalue is **complete** or has **no defect** if the **algebraic multiplicity** equals the **geometric multiplicity**.

Draw the phase diagram for

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



However, if the eigenvalue does not have enough independent eigenvectors (we say it is **defective** or **incomplete**) then we need to do more work.

I will show you the method for finding more independent solutions if the system is defective after I introduce the exponential of a matrix.