Matrix Exponential (3.8)

Definition If A is an n x n matrix, then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

if we define $A^0 = I$ (where I is the identity matrix) even if A is the zero matrix.

Fact: If $AB = BA$ then $e^A e^B = e^{A+B}$. This is not always true if A and B do not commute.

Corollary: e^A is invertible.

Proof: A and $-A$ commute, so $e^A e^{-A} = e^{A-A} = e^{[0]} = I$. Therefore e^{-A} is the inverse of e^A and so e^A is invertible.

But then the columns of A are always independent. $e^{At}\vec{C}$ is a linear combination of the columns of e^{At}. Prove that $e^{At}\vec{C}$ is a solution to $\vec{u}' = A\vec{u}$. Consequently, e^{At} is a fundamental matrix for the system and the columns form a basis for the solution set.

Example Write e^A as a 2 x 2 matrix if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Generalize this result to any n x n diagonal matrix.

How to find e^A if the eigenvalues of A are complete? We diagonalize: $A = X\Lambda X^{-1}$ where X has n column vectors equal to a basis of eigenvectors and Λ is the diagonal matrix of corresponding eigenvalues.

Example Find e^{At} if $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$.

Example Solve $\vec{u}\,' = A\vec{u}$ if $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$ by using e^{At} that was found in the last example. e^{At} is a (one of many) fundamental matrix for $\vec{u}\,' = A\vec{u}$.

Example Estimate e^B if $B = \begin{bmatrix} -0.3 & 0.1 \\ 0.2 & -0.2 \end{bmatrix}$ by using the first three terms of the Taylor series.

A matrix N is **nilpotent** iff there exists an integer n so that $N^n = [0]$. We can prove a matrix is similar to its Jordan Canonical form; this is enough to prove that a n x n matrix A with n repeated eigenvalues of λ implies $A - \lambda I$ is nilpotent. Since λI commutes with any matrix, we can write

$$e^{At} = e^{\lambda I t} e^{(A-\lambda I)t} = e^{\lambda t} I e^{(A-\lambda I)t} = e^{\lambda t} e^{(A-\lambda I)t}$$

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Use this information to quickly find e^{At} if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and then use e^{At} to solve $\vec{u}\,' = A\vec{u}$.