Repeated Roots with Defect (3.7)

Example Solve $\vec{u}' = A\vec{u}$ if $A = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$ by using $e^{At} = e^{\lambda t}e^{(A-\lambda I)t}$.

This is not the method used in most textbooks, and so your fundamental set will be different from the book answers. There is a good reason for this: the method above will not generalize to systems larger than 2 x 2. The standard method completes the fundamental set by finding a **generalized eigenvector** $\vec{v}^*$ so that $(A-\lambda I)\vec{v}^* = \vec{v}$ where $\vec{v}$ is an eigenvector of the repeated eigenvalue. Then since $(A-\lambda I)\vec{v} = \vec{0}$, $(A-\lambda I)^2\vec{v}^* = \vec{0}$, and so $(A-\lambda I)^n\vec{v}^* = \vec{0}$ for all integers $n \geq 2$. Consequently a new independent solution is,

$$e^{At}\vec{v}^* = e^{\lambda t}\left(I + (A-\lambda I)t + (A-\lambda I)^2\frac{t^2}{2} + ...\right)\vec{v}^* = e^{\lambda t}\left(\vec{v}^* + \vec{v}t\right)$$

Solve the system again using this method.

Solve $\vec{u}\,' = A\vec{u}$ if $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Solve $\vec{u}' = A\vec{u}$ if $A = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$.

Solve $\dfrac{d\vec{u}}{dt} = A\vec{u}$ if $A = \begin{bmatrix} 5 & 4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Solve $\dfrac{d\vec{u}}{dt} = A\vec{u}$ if $A = \begin{bmatrix} 5 & 4 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$.