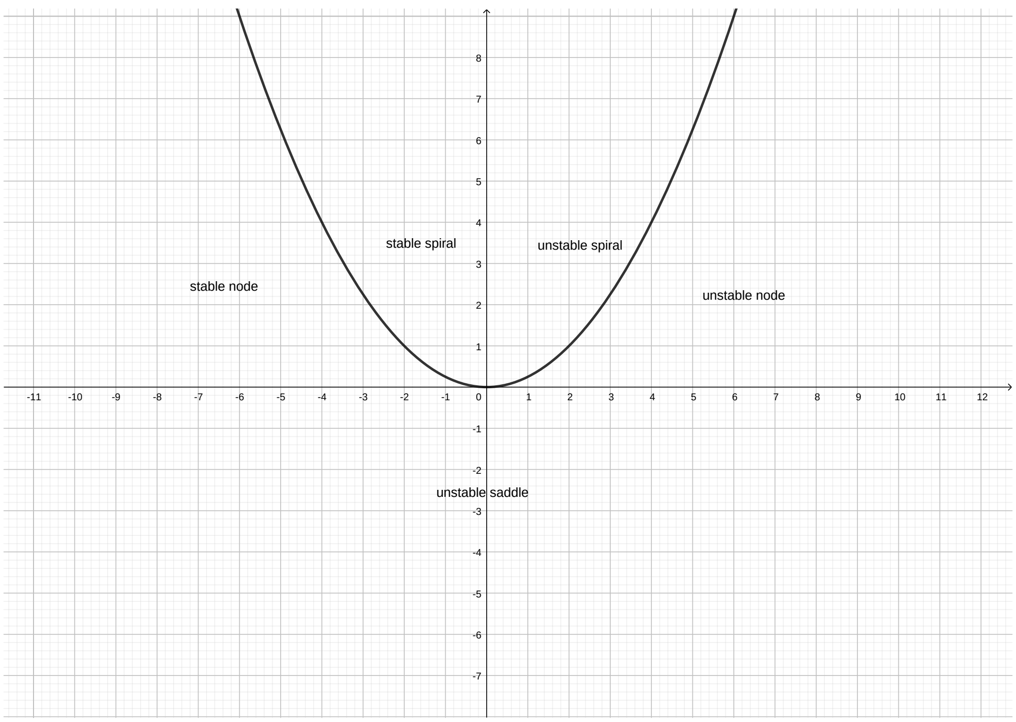
Nonlinear Autonomous Systems (Chapter 8)

$\vec{u}\,'(t) = \vec{f}(x_1, x_2, ..., x_n)$ where $\vec{u}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}$; that is, there are no independent variables on the right side. We have already dealt with linear autonomous systems, $\vec{u}\,' = A\vec{u}$, but nonlinear autonomous systems are a mystery - that's a shame, because they describe weather and other important phenomenon that are sensitive to initial conditions. All nonautonomous systems can be written as an autonomous system if we introduce enough new independent variables.

Example Write $x''(t) - x(t) + x^3(t) = t^3$ as an autonomous system and find all critical points.

Our goal is to determine the long term behavior of such systems. Our tactic will be to estimate a nonlinear system using a linear one at the critical points, much like we estimate a function with a taylor polynomial near a point. Below is a summary of what we already know about linear autonomous systems using the curve $\tau^2 = 4\Delta$ and recalling eigenvalues for a 2 x 2 system are $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$.



Border cases [1]

1) Stable **Center** if $\Delta > 0$, $\tau = 0$.

2) Unstable with unknown type if $\Delta = 0$, $\tau > 0$.

3) Stable with unknown type if $\tau^2 = 4\Delta$, $\tau < 0$.

4) Unstable with unknown type if $\tau^2 = 4\Delta$, $\tau > 0$.

5) Unknown stability and type if $\Delta = 0$, $\tau \leq 0$.

[1] If the critical point corresponds to an unknown type or stability, we need the more powerful Liapunov functions, or the "Generalized Stability Theory" that is based on the SVD. An automation for this last theory has been created: see **A Framework for the Automation of Generalized Stability Theory** by Rarrell, Cotter, and Funke. This material is beyond the scope of our course.

Example Find the critical points and the linearization at each critical point for $\begin{cases} x' = 2x - y^2 \\ y' = xy - y \end{cases}$.

Example Classify each critical point for $\begin{cases} x' = 2x - y^2 \\ y' = xy - y \end{cases}$. Sketch the phase plane for this system near each of the critical points.

Example Find and classify the critical points for $\begin{cases} x' = x^2 - y^2 \\ y' = xy - 4 \end{cases}$.