Nonhomogeneous Systems of Equations (3.9)

Using the proof used for the nonhomogeneous second order differential equations, we can show

$$\vec{u}\,'(t) = A\vec{u}(t) + \vec{f}(t) \implies \vec{u} = \vec{u}_c + \vec{u}_p$$

We can find $\vec{u}_p$ using a Guess and Check method (also called "undetermined coefficients") or using variation of parameters.

Guess and Check works well if $\vec{f}(t)$ is a constant vector $\vec{K}$ or an exponential multiple $e^{(a+bi)t}\vec{K}$.

Solve $\vec{u}\,'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{u}(t) + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

Solve $\vec{u}\,'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{u}(t) + e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Variation of Parameters

We assume we have a basis for the homogeneous solution to $\vec{u}\,'(t) = A\vec{u}(t) + \vec{f}(t)$, say

$$\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$$

Then the corresponding fundamental matrix is

$$\Phi = \left[\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n\right]$$

and

$$\vec{u}_c = \Phi\vec{C} \implies \vec{u}_c\,' = A\vec{u}_c \implies \Phi'\vec{C} = A\Phi\vec{C}$$

for any vector $\vec{C}$.

We then vary $\vec{C}$ to get a particular solution, $\vec{u}_p = \Phi\vec{v}$. Substituting back into the equation gives

$$\Phi'\vec{v} + \Phi\vec{v}\,' = A\Phi\vec{v} + \vec{f}(t)$$

which implies

$$\Phi\vec{v}\,' = \vec{f}(t)$$

We solve for the components of $\vec{v}$ using Cramer's rule or inverses, or elimination.

An example follows on the next page.

Solve $\vec{u}\,'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \vec{u}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$ if $\vec{u}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.