Cauchy-Euler Equations (Not in book - notes only.)

This type of equation shifts our attention to equations with varying coefficients. Bessel functions will eventually be defined from power series solutions to the Bessel differential equations; the Cauchy-Euler equations can be thought of as a degenerate form of the Bessel equation. Equations of the form

$$ax^2y'' + bxy' + cy = f(x)$$

are called Cauchy-Euler Equations. Write the equation as a system of equations with unknowns of y and y'.

Thus the Existence-Uniqueness theorem does not apply for $x = 0$; we say that $x = 0$ is a **singular point** of the equation. As a result multiple solutions (or no solution!) could go through the y-axis.

In each term of a Cauchy-Euler equation, the derivative reduces powers of x by 1, but then we multiply by x to get it back. This motivates a guess for an homogeneous solution: $y = x^r$.

Solve $x^2y'' + xy' - y = 0$. What solutions pass through the y-axis?

Substitute e^t for x into $ax^2y'' + bxy' + cy = f(x)$ to get $a\ddot{y} + (b-a)\dot{y} + cy = f(e^t)$ where $\dot{y} = \dfrac{dy}{dt}$ and $\ddot{y} = \dfrac{d^2y}{dt^2}$. Both equations have the same characteristic equation: $r^2 + (b-a)r + c = 0$.

Proving this requires showing $\dot{y} = xy'$ and $\ddot{y} = \dot{y} + x^2y''$. This is easy using the chain rule and $\dfrac{dx}{dt} = e^t = x$. Then $y(t) = y(\ln(x))$: this helps us understand the form of the solutions when the roots of the characteristic equation are repeated or are complex.

Write the general solution for $ax^2y'' + bxy' + cy = 0$ if:

a) r = 2 twice

b) r = $5 \pm 4i$

You may vary parameters or guess and check using x or t.

Solve $2x^2y'' - 10xy' + 16y = 16x^6$, $y(1) = 0$ and $y'(1) = 0$.

As time allows.

Be sure to put the equation in standard form first if varying parameters to determine the correct forcing function! That means the leading coefficient must be one before we can read-off the forcing function.

Solve $x^2y'' + xy' - y = \dfrac{x}{x+1}$.