Series Solutions (Chapter 7)

If there are no constraints on the independent variable, we can guess the solution is of the form $y = \sum_{n=0}^{\infty} c_n x^n$.

Solve $y'' + 2xy' + 2y = 0$.

Notice that $y'' + 2xy' + 2y = 0$ can be written as $\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -2x \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$ which satisfies the hypotheses of the existence-uniqueness theorem everywhere, and in particular at $x = 0$. We say $x = 0$ is an **ordinary** point. In this example every point is an ordinary point. Notice that $q(x) = 2x$ and $r(x) = 2$ have continuous derivatives of all orders; in this case, we say the $q(x)$ and $r(x)$ are **analytic**.

Definition: $x = a$ is an **ordinary point** of $p(x)y'' + q(x)y' + r(x)y = 0$ if $\dfrac{q(x)}{p(x)}$ and $\dfrac{r(x)}{p(x)}$ are analytic in a box about $x = a$.

Theorem: If $x = a$ is an **ordinary point** of $p(x)y'' + q(x)y' + r(x)y = 0$, then there exist two linearly independent power series solutions about $x = a$ of the form $y = \displaystyle\sum_{n=0}^{\infty} c_n (x - a)^n$.

Definition: If $x = a$ is **not** an ordinary point of $p(x)y'' + q(x)y' + r(x)y = 0$ then $x = a$ is a **singular point** of the equation.

$Ax^2y'' + Bxy' + Cy = 0 \implies \dfrac{q(x)}{p(x)} = \dfrac{B}{Ax}$ and $\dfrac{r(x)}{p(x)} = \dfrac{C}{Ax^2}$, neither of which are analytic at zero. So $x = 0$ is a singular point. All the other values of x **are** ordinary points. The Cauchy-Euler equation was easy to solve using $y = x^r$, however, so we will call $x = 0$ a **regular singular point**.

Definition: Suppose $p(x)$, $q(x)$, and $r(x)$ are analytic. Then $x = a$ is a **regular singular point** of $p(x)y'' + q(x)y' + r(x)y = 0$ if $\displaystyle\lim_{x \to a} (x - a)\frac{q(x)}{p(x)}$ and $\displaystyle\lim_{x \to a} (x - a)^2 \frac{r(x)}{p(x)}$ both exist and are finite.

Find all singular points of $x^3(x - 2)^4 y'' + x(x - 2)^3 y' + (x - 2)^5 y = 0$ and classify them as regular or irregular.

Solve $xy'' + 0.5y' - 6y = 0$. $x = 0$ is a regular singular point and the equation is similar to a Cauchy-Euler equation if we multiply by x, so we are tempted to try a guess of $y = x^r$. However, the last term makes that inaccurate, so we multiply our guess by a power series. We will use the **Frobenius Method** and guess

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

where we assume $r \in \mathbb{R}$ is chosen so that $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

Now substitute and find an **indicial** equation.

Once the indicial equation is found, we proceed as we did for an ordinary point, getting one recursion formula for each index.

Frobenius Theorem: If $x = a$ is a regular singular point of $p(x)y'' + q(x)y' + r(x)y = 0$ then there is at least one nonzero solution of the form $y = \sum_{n=0}^{\infty} c_n(x-a)^{n+r}$ for some $r \in \mathbb{R}$ and $c_0 \neq 0$.

Bessel Functions

Bessel functions of order $p \in \mathbb{R}$ solve the Bessel equations of order p

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

Notice how similar this equation is to Cauchy-Euler equations; in particular, $x = 0$ is a regular singular point.

Use the Frobenius method to find a solution for $p = 3$ about $x = 0$.

If we multiply that solution by $\frac{1}{2^3 \cdot 3!}$ then we get another solution called the **Bessel function of the first kind with order 3**. The notation is $J_3(x) = \frac{1}{2^3 \cdot 3!}y$. We still need another independent solution, the **Bessel function of the second kind with order 3**, $Y_3(x)$, to complete a basis for the solution set. We do it this way:

If p is not an integer, then J_{-p} is another independent solution found using the Frobenius method. We only have this problem of one solution if p is an integer. So we define

$$Y_p(x) = \frac{\cos(p\pi)J_p(x) - J_{-p}(x)}{\sin(p\pi)}$$

if p is **not** an integer and

$$Y_n(x) = \lim_{p \to n} Y_p(x)$$

if n is an integer. $Y_p(x)$ is a solution to the differential equation because it is a linear combination of two solutions, and independent from $J_p(x)$ since $J_{-p}(x)$ is. Once we prove $Y_n(x)$ is a solution to the differential equation and is not dependent on $J_n(x)$, we can write the solution for $x^2y'' + xy' + (x^2 - p^2)y = 0$ as

$$y = c_1 J_p(x) + c_2 Y_p(x)$$

for **any** value of p whether it is an integer or not. That proof is outside the scope of this class.

Write the solution to $x^2y'' + xy' + (x^2 - \frac{9}{4})y = 0$ without doing any other work.

Use a substitution to find the solution to $x^2y'' + xy' + (4x^2 - 9)y = 0$.

Theorem: If the indicial roots r_1 and r_2 from the Frobenius method do not differ by an integer, then the method produces two independent solutions.

As time allows: an aging spring.

$\ddot{x}(t) + e^{-0.02t}x(t) = 0$ represents an aging spring with constant decreasing as time increases. Let's solve it.

First let $v = 100e^{-0.01t}$ and show that $\frac{dx}{dt} = \frac{dx}{dv}\left(\frac{-v}{100}\right)$.

Next show that $\ddot{x}(t) = \frac{d^2x}{dv^2}\left(\frac{-v}{100}\right)^2 - 0.01\frac{dx}{dv}\left(\frac{-v}{100}\right)$.

Substitute to get a Bessel equation of order zero and write the solution.

If even more time allows:
Solve $2x^2y'' + x(2x+1)y' - y = 0$ using the Frobenius method.